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Rearrangements of functions, variational problems and elliptic equations for vortices

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Summary

The results of my thesis concern the existence theory for certain variational problems, which are related to vortex flows of ideal fluids. The method that is used, is similar to the one that was proposed by Benjamin [5] and motivated recent work by Burton [10], in which vortex rings can be obtained as maximisers of a functional that is related to the kinetic energy over the set of rearrangements of a fixed function.

We start out in the first Chapter by explaining some preliminary concepts in our work (rearrangements of functions and existence theorems for steady vortices). We then explain Burton's method for maximisation of functionals and briefly state of the main results of this thesis.

In the second Chapter, we develop the existence theory of a variational problem similar to the one governing steady 2-dimensional ideal fluid flows containing symmetric vortex pairs. A functional related to the kinetic energy and the "generalised impulse" I_n corresponding to the parameter $n > 0$, is shown to attain a maximum value relative to the set of rearrangements of a prescribed function. Specifically, if λ is a parameter corresponding to the strength of the background flow at infinity, then we show that there exists a maximiser among flows whose vortices are rearrangements of a prescribed function, for all $\lambda > 0$ when $n \geq 3$ and for only small $\lambda > 0$ if $n = 2$, where this last case represents the existence of vortex pairs in a "two phase" shear flow.

In the third Chapter, we adapt the method of Burton [18] to study a problem related to the one studied in Chapter 2. We maximise a functional that is related to the kinetic energy over the weak closure of the set of rearrangements of a prescribed function, and for which the "generalised impulse" I_n has a prescribed value. We prove that if $n \geq 3$, then for any I_n, the constrained maximiser is a rearrangement of the prescribed function, and if $n \in \{1, 2\}$, then the maximiser is a rearrangement only for I_n large.

In the last Chapter, we prove the existence theory for a slightly different variational problem, governing a steady 3-dimensional ideal fluid flow containing axisymmetric steady vortex rings. The method that is used here is similar to that used in Chapter 2. If λ is the value of the strength of a steady ideal fluid flow at infinity, and I_{2n} is the "generalised impulse", then we prove that for all $\lambda > 0$ and $n \geq 4$, a functional related to the kinetic energy and the "generalised impulse" I_{2n}, is attained a maximum value relative to the set of rearrangement of a prescribed function. We also show that by making an assumption which we believe to be true, in the case when $n \in [2, 4[$ and λ is small, the same functional is attained a maximum value relative to set of rearrangements of prescribed function for all small $\lambda > 0$. The case when $n = 2$ represents the existence of a steady vortex ring in Poiseuille flow.

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Contents

Chapter 1

Introduction

The theory of maximisation of a convex functional over the set of rearrangements of a fixed function has an important role in some physical problems, particularly in Benjamin's theory [5] for vortex rings in an ideal fluid in three dimensions. In this theory, it was claimed that a steady vortex ring can be characterised by maximising some functional related to the kinetic energy, over the set of rearrangements of a fixed function. In Benjamin's approach, the rearrangements represent possible configurations for a vortex in the flow. Later on, Burton [10] developed the required functional analysis supporting this theory. In this thesis, Burton's theory has been applied to investigate two variational problems, that are connected with vortex rings in three dimensions, and with symmetric vortex pairs in two dimensions.

Before outlining our main results, we give some brief background of rearrangements of a function and maximisation of a convex functional. Also we survey the relevant literature.

1.1 Rearrangements of functions

Let (Ω, Σ, μ) and (Ω', Σ', μ') be two positive measure spaces with $\mu(\Omega) = \mu'(\Omega')$, where Σ and Σ' are two σ-algebra on Ω and Ω' respectively. A real measurable function $f \geq 0$ on Ω is a rearrangement of a real measurable function $g \geq 0$ on Ω', or g is a rearrangement of f, if

$$\mu(\{x \in \Omega | f(x) > \alpha\}) = \mu'(\{x \in \Omega' | g(x) > \alpha\}) \tag{1.1}$$

for all $\alpha > 0$, and both sides in (1.1) are finite. In this case, if $f \in L^p(\mu)$ where $1 \leq p < \infty$, then it follows that $\|f\|_p = \|g\|_p$; hence $g \in L^p(\mu')$. For a non-negative measurable function f_0 on Ω, $\mathcal{F}(f_0)$ will be used to denote the set of rearrangements of f_0 on Ω.

If $f_0 \geq 0$ is defined on the half-line $[0, \infty)$, and the measure is Lebesgue measure, Eydeland, Spruck and Turkington [23] characterised the set $\mathcal{F}(f_0)$ as

$$\mathcal{F}(f_0) = \left\{ f \geq 0, \int_0^\infty (f - \alpha)^+ dx = \int_0^\infty (f_0 - \alpha)^+ dx \quad \forall \alpha > 0 \right\},$$

where $+$ denotes the positive part of the function.

Let f be a non-negative measurable function on $\mathbb{R}^n$ such that $\mu_n(f^{-1}((\alpha,\infty)) < \infty$ for all $\alpha > 0$, where μ_n is the n-Lebesgue measure in $\mathbb{R}^n$ $(n \geq 1)$. Here, we list some well known examples of special rearrangements of f.

Definition 1.1. *The symmetric decreasing rearrangement of f, that is denoted by $f_\triangle$ is given by*

$$f_\triangle(t) := \begin{cases} \max\{\alpha > 0 |\ F(\alpha) \geq 2|t|\} & \text{if there is such } \alpha, \\ 0, & \text{otherwise} \end{cases}$$

for all $t \in \mathbb{R}$, where $F(\alpha) := \mu_n(f^{-1}((\alpha,\infty))$ is the distribution function of f.

Definition 1.2. *The Steiner-symmetrisation of f about the plane $x_n = 0$, that is denoted by f^s is defined by*

$$f^s(x_1,\cdots,x_n) := f_\triangle(x',.), \quad \text{for all} \quad (x_1,\cdots,x_n) \in \mathbb{R}^n$$

where $x' = (x_1,\cdots,x_{n-1})$. Note that f^s is a rearrangement of f.

Definition 1.3. *The spherically decreasing rearrangement f^* of f, also called the Schwarz symmetrisation, is defined by*

$$f^*(x) := \max\{\alpha > 0 |\ F(\alpha) \geq |x|^n \omega_n\},$$

where ω_n denotes the volume of unit ball in $\mathbb{R}^n$ and F is the distribution function as in Definition 1.1.

From Definition 1.3, Pólya and Szegö [36] showed that if $f \in W^{1,2}(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} |\nabla f^*|^2 d\mu_n \leq \int_{\mathbb{R}^n} |\nabla f|^2 d\mu_n. \tag{1.2}$$

In the case of equality, Brothers and Ziemer [8] showed that if $\mu_n(\{f^*(s) > \alpha\})$ is absolutely continuous, then f is almost everywhere equal to a translate of f^*. Burton and McLeod [15] used these results to study maximisation and minimisation of certain weakly continuous convex functionals.

Let f, g and h three non-negative measurable functions in $\mathbb{R}^n$ $(n \geq 1)$ such that for all $\alpha > 0$ the distribution functions $F(\alpha)$, $G(\alpha)$ and $H(\alpha)$ corresponding to f, g and h respectively are finite. In the case when $n = 1$ Riesz [37] showed the following inequality holds:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(x-y)h(y)dxdy \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^*(x)g^*(x-y)h^*(y)dxdy. \tag{1.3}$$

If the left-hand side equals ∞, then the right-hand side equals ∞. Later on, Brascamp, Lieb and Luttinger [6] generalised (1.3) for any $n \geq 1$. The reader will find more details about rearrangements inequalities in Lieb and Loss [31].

Definition 1.4. *Let $\Omega \subset \mathbb{R}^n$ be such that $\mu_n(\Omega) < \infty$, and let f be a non-negative measurable function defined on Ω. The (essentially) unique decreasing rearrangement of f is the function $f^{\triangle}$ defined by*

$$f^{\triangle}(t) := \max\{\alpha > 0 | \ F(\alpha) \geq t\}, \tag{1.4}$$

for all $0 < t < \mu_n(\Omega)$, where F is the distribution function of f as in Definition 1.1.

If f and g are two non-negative functions in $L^p(\Omega)$ and $L^q(\Omega)$ respectively, where $p \in [1, \infty)$ and q is the conjugate exponent of p, then the following inequality is classical

$$\int_{\Omega} fg d\mu_n \leq \int_0^{\mu_n(\Omega)} f^{\triangle} g^{\triangle} d\mu_1; \tag{1.5}$$

for a proof in this general setting, see Burton [10].

Definition 1.5. *Let f and g be two non-negative functions defined on $(0, \infty)$. We say g is a curtailment of f at $l \in [0, \infty]$ if*

$$g = 1_{(0,l)} f^{\triangle}$$

and g is a rearrangement of a curtailment of f at $l \in [0, \infty]$ if $g^{\triangle}$ is curtailment of f at some $l \in [0, \infty]$.

Now let U be an unbounded domain in $\mathbb{R}^n$ $(n \geq 1)$ of infinite Lebesgue μ_n measure. Let $T : (0, \infty) \to U$ be a measure preserving transformation. Define a map $M : L^p(U) \to L^p(0, \infty)$ $(p \geq 1)$ by

$$M(f) = f \circ T.$$

For a non-negative f and g in $L^p(U)$ we have

$$g \in \mathcal{F}(f) \quad \text{if and only if} \quad M(g) \in \mathcal{F}(M(f)).$$

This concept together with Definition 1.5 show that if $f_0 \in L^p(U)$ $(1 < p < \infty)$ is a non-negative function, then $g \in L^p(U)$ is a rearrangement of a curtailment of f_0 if and only if $M(g)$ is a rearrangement of a curtailment of $M(f_0)$. The set of rearrangements of curtailments of f_0 will be denote by $\mathcal{RC}(f_0)$.

The concept of the weak closure of the set $\mathcal{F}(f_0)$ of a given function $f_0 \geq 0$ has been developed by Douglas [20, 21]. For a given non-negative function $f_0 \in L^p(U)$, where U is a domain in $\mathbb{R}^n$ $(n \geq 1)$ of infinite measure, he showed that the weak closure of $\mathcal{F}(f_0)$ is given by

$$\mathcal{W}(f_0) = \left\{ f \geq 0, \ f \ \text{ measurable on } U, \int_U (f - \alpha)^+ d\mu_n \leq \int_U (f_0 - \alpha)^+ d\mu_n \ \forall \alpha > 0 \right\}.$$

Douglas also proved that $\mathcal{W}(f_0)$ is convex and weakly sequentially compact, and in the terms of the Definition 1.5, he proved that the set of extreme points of $\mathcal{W}(f_0)$ is $\mathcal{RC}(f_0)$. In the case of non-negative $f_0 \in L^1(I)$, where I is the unit interval of $\mathbb{R}$, Ryff [38] characterised the weak closure of the set of rearrangements of f_0 as

$$\left\{ f \in L^1(I) \middle| \int_0^s f^{\triangle} dx \le \int_0^s f_0^{\triangle} dx \ \text{ for } \ s \in (0,1), \ \ f \ge 0 \ \text{ and } \ \|f\|_1 = \|f_0\|_1 \right\}.$$

Ryff proved also that the weak closure of the set of rearrangements of f_0 is convex, and equal to the closed convex hull of the set of rearrangements. Brown [9] generalised this result; he proved that if $f_0 \in L^p(I)$ where $1 < p < \infty$, then this result is still true. Burton and Ryan [14] showed that the intersection of the weak closure of the set of rearrangements with a set of finitely many closed affine hyperplanes is equal to the convex hull of the set of rearrangements intersected with the hyperplanes.

1.2 Existence theorems for steady vortices

We begin this section with a discussion of steady vortex-rings in $\mathbb{R}^3$, a description of Benjamin's theory [5], and a survey of the literature. We conclude with an account of the analogous theory for steady planar vortices. Let (r, θ, z) be cylindrical coordinates in $\mathbb{R}^3$. Consider axisymmetric ideal fluid flow in $\mathbb{R}^3$; incompressibility guarantees the existence of a Stokes stream function $\Gamma(r, z)$ for the flow, which means the velocity V satisfies

$$\mathrm{V} = \left(-\frac{1}{r}\frac{\partial \Gamma}{\partial z}, 0, \frac{1}{r}\frac{\partial \Gamma}{\partial r}\right).$$

Then the vorticity is given in terms of the velocity by

$$\mathrm{curl}\ \mathrm{V} = (0, r\mathcal{L}\Gamma, 0),$$

where

$$\mathcal{L}\Gamma = -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Gamma}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 \Gamma}{\partial z^2}. \tag{1.6}$$

The region where $r\mathcal{L}\Gamma = v \neq 0$ is called the vortex core. If Γ satisfies the equation

$$\mathcal{L}\Gamma = \phi \circ \Gamma, \tag{1.7}$$

where ϕ is unknown function, then the flow is steady.

The existence problem of steady vortex rings is to show that the non-linear equation

$$\mathcal{L}\Psi = \phi \circ \left(\Psi - \frac{\lambda}{2} r^2\right) \ \text{ in } \ \Pi \tag{1.8}$$

has a solution with respect to the boundary conditions

$$(BC) \quad : \quad \Psi(0, z) = 0\ , \ \Psi(r, z) \to 0 \ \text{ and } \ |\nabla \Psi| \to 0 \ \text{ as } \ r^2 + z^2 \to \infty,$$

where Π is the half-plane that is defined by $r > 0$ and $-\infty < z < \infty$, λ is a positive constant corresponding to the speed that the flow approaches at infinity and $\phi : \mathbb{R} \to \mathbb{R}$ is an unknown increasing function. Hence by (1.7), $\Gamma(r, z) = \Psi(r, z) - \frac{\lambda}{2}r^2$ is the Stokes stream function for the flow

The existence theory of steady vortex rings has been the subject of work by a number of authors. Nearly three decades ago, it was addressed by Benjamin [5]. His approach was concerned with seeking a solution of (1.8) with same boundary conditions (BC), for which v/r is a rearrangement of a prescribed function and for which a value is prescribed for either the speed λ at infinity, or the impulse $I(v)$, which for a fluid of unit density is given by

$$I(v) = \int_{\mathbb{R}^3} rv.$$

The method Benjamin [5] proposed to achieve the existence of a solution is based on solving one of the following variational problems

$$\max_{v \in \mathcal{F}(v_0)} (E(v) - \lambda I(v)), \tag{1.9}$$

or

$$\max_{I(v)=I, v \in \mathcal{F}(v_0)} E(v), \tag{1.10}$$

where $E(v)$ is a functional related to the kinetic energy, $\mathcal{F}(v_0)$ is the set of rearrangements of a given function $v_0 \geq 0$ and I is a positive number. One of the facts motivating this approach is that $I(v)$ and the volumes of the sets $\{v/r \geq \alpha\}$ (for each $\alpha > 0$) are preserved in axisymmetric motions of an ideal fluid in $\mathbb{R}^3$, so the quantities $I(v)$ and the prescribed function v_0 are physically meaningful.

Burton [10] applied his theory to study equation (1.8) in a bounded axisymmetric domain $\Omega \subset \mathbb{R}^3$. By studying the variational problem (1.9) (which describes the existence of steady vortex rings with a prescribed λ), he showed that for each λ there exists $v_\lambda \in \mathcal{F}(v_0)$ that maximises the functional $E(v) - \lambda I(v)$ over $\mathcal{F}(v_0)$, and if we set $\Psi := Kv_\lambda$, then Ψ satisfies the equation (1.8) almost everywhere in Ω, where Kv_λ is the weak solution of $\mathcal{L}\Psi = v$ belonging to $\mathcal{H}$, the Hilbert space obtained by completing $\mathcal{D}(\Omega)$ with the scalar product

$$\langle u, v \rangle_H = \int_{\Omega} \frac{2\pi}{r} \nabla u . \nabla v dr dz. \tag{1.11}$$

For the second variational problem (1.10), Burton proved that if I satisfies a certain feasibility condition, then there exists $v \in \mathcal{F}(v_0)$ that maximises the functional $E(v)$ subject to $I(v) = I$ and $v \in \mathcal{F}(v_0)$ (the existence of steady vortex rings with a prescribed impulse), and if $\Psi := Kv$, then Ψ satisfies (1.8) almost everywhere in Ω, for some λ and some increasing function ϕ. In each cases $\Psi - \frac{\lambda}{2}r^2$ is the Stokes stream function for the flow.

Recently, the problem of the existence of steady vortex rings has been studied by Badiani and Burton [4] and Badiani [2], where they showed that the equation (1.8) with the boundary condition (BC) has a solution for which v/r belongs to $\mathcal{W}(v_0)$, the weak closure of the set $\mathcal{F}(v_0)$. The two authors found that for each $\lambda > 0$, there exists v_λ that maximises the functional $E(v) - \lambda I(v)$ over $\mathcal{W}(v_0)$. Additionally, if we set $\Psi := Kv_\lambda$, then Ψ satisfies equation (1.8) almost everywhere in Π for some increasing function ϕ. Furthermore, they showed that if the prescribed function v_0 is positive and constant on its support, then v_λ is a rearrangement of v_0 on Π unless $v_\lambda \equiv 0$. For a prescribed impulse $I(v)$, Burton [18] showed that the functional E can be maximised subject to $I(v) = I > 0$ and $v \in \mathcal{W}(v_0)$. For the cylindrical domain $\Omega \subset \mathbb{R}^3$ defined by $-\infty < z < \infty$ and $0 < r < R$ (R constant), Douglas [20] showed that for all $\lambda > 0$, the functional $E(v) - \lambda I(v)$ attains a maximum value relative to $\mathcal{W}(v_0)$, and every maximiser v_λ belongs to $\mathcal{RC}(v_0)$, the set of rearrangements of curtailments of prescribed function v_0. Additionally, if the prescribed function $v_0 \in L^1 \cap L^p$ where $p > 5/2$, then $\Psi := \widetilde{K}v_\lambda$ satisfies the equation (1.8) almost everywhere in the domain Ω, where $\widetilde{K}v_\lambda$ is the weak solution of $\mathcal{L}\Psi = v$ belonging to $\mathcal{H}$, the Hilbert space which is obtained by completing $\mathcal{D}(\Omega)$ with the scalar product defined in (1.11). In this case also, Burton [12] showed that if λ is small enough and positive, then the functional $E(v) - \lambda I(v)$ attains a maximum value relative to $\mathcal{F}(v_0)$. If v_λ is a maximiser and $\Psi := \widetilde{K}v_\lambda$ the weak solution for $\mathcal{L}\Psi = v$, then Ψ satisfies the equation (1.8) almost everywhere in Ω, for some increasing function ϕ.

For different points of view, Fraenkel and Berger [24] proved the existence of steady vortex rings by proving the existence of a solution for the non-linear equation

$$\mathcal{L}\Psi = k\phi \circ (\Psi - \frac{\lambda}{2}r^2 - \gamma) \quad \text{in} \quad \Pi \tag{1.12}$$

with respect to (BC), where $k > 0$, $\gamma \geq 0$, $\lambda > 0$ are prescribed and ϕ is a prescribed non-decreasing Hölder continuous function. Fraenkel and Berger achieved this solution by maximising a certain functional on the surface of a sphere in a Sobolev space with energy norm. In the case when $\gamma = 0$ and ϕ is the Heaviside function ($\phi(t) = 0$ if $t \leq 0$ and $\phi(t) = 1$ if $t > 0$), Fraenkel and Amick [25] showed that any solution in $\mathcal{H}$ for (1.12) is equal, modulo translation in the z-direction, to the explicit "spherical" solution found by Hill [28], where $\mathcal{H}$ is the Hilbert space obtained by completing $\mathcal{D}(\Pi)$ with the scalar product (1.11). By using the contraction mapping theorem, Norbury [33] proved existence of a family of steady vortex rings close to Hill's spherical vortex ring. Ambrosetti and Struwe [1] used a minimax method to construct a solution for the equations (1.12) with prescribed k, γ and $\lambda > 0$. The study of Ni [32] may be regarded as an extension of the work of Fraenkel and Berger [24]; he used a minimax principle to show that such a functional has a critical point which gives rise to an existence theorem for vortex rings. Friedman and Turkington [26] showed that the existence of steady vortex rings is given by studying a variational problem in which the impulse $I(v) = 1$ and the essential supremum of the vorticity is less than or equal to a prescribed constant.

In a 2-dimensional ideal fluid, any flow is given by a stream function $\psi(x_1, x_2)$ related to the velocity ω by

$$\omega = (\frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1});$$

hence the vorticity is given by

$$-\Delta \psi := \zeta,$$

where Δ is the Laplacian operator in 2-dimensions. Burton [13] studied the existence of a steady symmetric vortex pairs, by using Benjamin's approach for vortex rings, in which the existence of steady vortex can be characterised by solving one of the variational problems (1.9) or (1.10). The kinetic energy $E(\zeta)$ and the impulse $I(\zeta)$ are here given by

$$E(\zeta) = \frac{1}{2} \int_{\Pi} \zeta(x) K\zeta(x) dx \tag{1.13}$$

and

$$I(\zeta) = \int_{\Pi} x_2 \zeta(x) dx,$$

where $dx = dx_2 dx_1$, Π the half-plane in $\mathbb{R}^2$ defined by $x_2 > 0$ and K is the inverse of $-\Delta$ with homogeneous Dirichlet boundary condition on Π. Burton proved that if λ is small and positive and subject to restrictions on ζ_0, then the functional $E(\zeta) - \lambda I(\zeta)$ attains a maximum value relative to $\mathcal{F}(\zeta_0)$. He also showed that if ζ_λ is a maximiser for small λ, and $\psi := K\zeta_\lambda$, then ψ satisfies the semi-linear elliptic equation

$$-\Delta \psi = \phi \circ (\psi - \lambda x_2) \tag{1.14}$$

almost everywhere in Π, for some increasing function ϕ. Burton also proved that the problem analogous to (1.10) has a solution by showing that there exists $I_0 > 0$ such that if $I(\zeta) > I_0$, then the functional $E(\zeta)$ attains its supremum subject to $\mathcal{F}(\zeta_0)$ and $I(\zeta) = I$ and if $\psi := K\zeta$ for a solution ζ, then Ψ satisfies (1.14) almost everywhere in Π, for some positive λ and an increasing function ϕ. Similarly to this problem, Badiani [3] proved the existence theorem for a steady planar flow past an obstacle, containing a symmetric vortex pair and approaching a uniform flow at infinity, by combining the methods of Burton [10] and Turkington [39, 40]. The latter showed the existence of vortex pairs in flows occupying the whole space $\mathbb{R}^2$ or $\mathbb{R}^2 \backslash D$, where D is bounded, simply connected, symmetric in the x_1-direction, containing the origin in its interior and having smooth boundary. Turkington found these results by using the maximisation of the kinetic energy over a different set of admissible functions. Norbury [34], using analogous methods to Fraenkel and Berger [24] proved the existence of steady vortex pair satisfying (1.14) with prescribed ϕ. Based on minimising a functional corresponding to the vortex strength parameter, Jianfu [29] studied the existence and asymptotic behaviour of planar vortex pairs.

In the 2-dimensional space $\mathbb{R}^2$, Nycander [35] studied the existence theory of a steady vortex in a shear flow by using Benjamin's approach [5]. He maximised the functional $E(\zeta) - \lambda I_2(\zeta)$ over $\mathcal{F}(\zeta_0)$ where ζ_0 is a given function vanishing outside a set of finite measure, that satisfies $0 < m < \zeta_0 < M < \infty$, and

$$I_2(\zeta) = \frac{1}{2}\int\limits_{\mathbb{R}^3} x_2^2\zeta(x)dx.$$

Here $E(\zeta)$ is defined by (1.13) but $K\zeta$ is the Newtonian potential of ζ. Nycander proved that for all $\lambda > 0$, the above functional attains a maximum value relative to $\mathcal{F}(\zeta_0)$, where ζ_0 satisfies the above conditions. Later on, Emamizadeh [22], using Burton's theory [10], showed that Nycander's result [35] remains true even if ζ_0 does not satisfy the above upper and lower bound, but belongs to L^p for suitable p.

1.3 Description of Burton's method

Here, we are interested in the theory of the maximisation of functional, which formed the main part of Burton's theory [10]. Henceforth, let $1 \leq p < \infty$ and q be the conjugate exponent of p. Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be such that $\mu_n(\Omega) < \infty$. Let $f_0 \in L^p(\Omega)$ be a non-negative function such that supp $f_0 \subset \Omega$, where supp f_0 is the support of f_0.

1.3.1 Maximisation of functionals

Burton [10] proved that if $g \in L^q(\Omega)$, then the functional

$$\langle f, g\rangle = \int\limits_{\Omega} f(x)g(x)d\mu_n$$

attains its supremum on the set $\mathcal{F}(f_0)$. Furthermore, he proved that if $\langle ., g\rangle$ has a unique maximiser relative to $\mathcal{F}(f_0)$, say $\widetilde{f}$, then there exists an increasing function $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\widetilde{f} = \phi \circ g$$

almost everywhere in Ω. Burton used these results to study the variational problem

$$\max_{v\in\mathcal{F}(f_0)} \Phi(v),$$

where $\Phi : L^p(\Omega) \to \mathbb{R}$ is a functional assumed to be convex and weakly sequentially continuous. [10, Lemma 6] shows that $\mathcal{W}(f_0)$ is weakly sequentially compact. Hence by using the Theorem of Weierstrass, Φ attains a maximum value, say at $\bar{f} \in \mathcal{W}(f_0)$. Since Φ is convex and strongly continuous, an application of the Theorem of Hahn-Banach shows that $\partial\Phi(f) \neq \emptyset$, where $\partial\Phi(f)$ denotes the sub-differential of Φ at any point $f \in L^p(\Omega)$. We choose $g \in \partial\Phi(\bar{f})$; it then follows that

$$\langle \bar{f}, g\rangle \leq k := \sup_{f\in\mathcal{F}(f_0)} \langle f, g\rangle.$$

By choosing $\widetilde{f} \in \mathcal{F}(f_0)$ such that

$$\langle \widetilde{f}, g\rangle = k,$$

which is possible by using the fact that the functional $\langle ., g\rangle$ attains a maximum value relative to $\mathcal{F}(f_0)$. Hence it follows that

$$\Phi(\widetilde{f}) \geq \Phi(\bar{f}) + \langle \widetilde{f} - \bar{f}, g\rangle \geq \Phi(\bar{f}).$$

Therefore $\widetilde{f}$ maximises Φ relative to $\mathcal{F}(f_0)$. Assume now that Φ is strictly convex. Let $f \in \mathcal{F}(f_0)$ be such that $\widetilde{f} \neq f$; then it follows that

$$\Phi(\widetilde{f}) \geq \Phi(f) > \Phi(\widetilde{f}) + \langle f - \widetilde{f}, g\rangle,$$

so

$$\langle f, g\rangle < \langle \widetilde{f}, g\rangle.$$

Thus, $\widetilde{f}$ is the unique maximiser of $\langle ., g\rangle$ relative to $\mathcal{F}(f_0)$, hence there exists an increasing function $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\widetilde{f} = \phi \circ g$$

almost everywhere in Ω. All the above stages can be summarised by the following Theorem:

Theorem 1.1. *Let Φ be a real strictly convex functional on $L^p(\mu)$, sequentially continuous in the weak topology. Then Φ attains a maximum value relative to $\mathcal{F}(f_0)$. If $\widetilde{f}$ is a maximiser and $g \in L^q(\mu)$ is a subgradient of Φ at $\widetilde{f}$, then $\widetilde{f} = \phi \circ g$ almost everywhere, for some increasing function ϕ.*

1.3.2 Applications

To illustrate some applications for the above results, let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be an unbounded domain that is Steiner-symmetric about the plane $x_n = 0$ such that $\mu_n(\Omega) < \infty$. Let K be an operator on function on Ω such that $K : L^p(D) \to L^q(D)$ is a strictly positive, symmetric and compact operator for every bounded set $D \subset \Omega$ that is Steiner-symmetric about the plane $x_n = 0$ and satisfies $\mu_n(\text{supp } f_0) \leq \mu_n(D)$. Moreover, we assume that K satisfies

$$\int_D v(x)Kv(x)d\mu_n \leq \int_D v^s(x)Kv^s(x)d\mu_n, \tag{1.15}$$

where v^s is the Steiner-symmetrisation of v about the plane $x_n = 0$. Define the functional $\Phi : L^p(\Omega) \to \mathbb{R}$ by

$$\Phi(v) = \int\limits_{\Omega} v(x)Kv(x)d\mu_n - \int\limits_{\Omega} u(x)v(x)d\mu_n,$$

where u is Steiner-symmetric on Ω and $u \in L^q(D)$ for each bounded set $D \subset \Omega$. We explain now a strategy to show that the functional Φ attains a maximum value relative to $\mathcal{F}(f_0)$. To do that, the functional Φ will first be shown, by using the inequality (1.15) and Theorem 1.1, to have a maximiser relative to $\mathcal{F}_D(f_0)$ that is Steiner-symmetric about the plane $x_n = 0$. Here $\mathcal{F}_D(f_0)$ is the set of all rearrangements of f_0 on D respect to μ_n measure. Furthermore, if v_D is any maximiser then v_D satisfies the equation

$$v_D = \phi \circ (Kv_D - u) \tag{1.16}$$

almost everywhere in D, for some increasing function ϕ. In order to pass to the unbounded domain Ω, by deriving some estimates for the function Kv_D, it will be shown that

$$\mu_n(\{x \in D | Kv_D(x) - u(x) > 0\}) \geq \mu_n(\text{supp } f_0).$$

By deriving further estimates, it will be shown also there exists a bounded domain $\widetilde{D}$ independent of D that is Steiner-symmetric about the plane $x_n = 0$, for which

$$\{x \in D | Kv_D(x) - u(x) > 0\} \subset \widetilde{D}.$$

Hence by using the fact ϕ is increasing function, then it follows that

$$\{x | v_D(x) > 0\} \subset \widetilde{D}.$$

Therefore v_D is a maximiser for Φ relative to $\mathcal{F}(f_0)$ if D is chosen so that $\widetilde{D} \subset D$.

1.4 Outline of main results

In the following Chapter, we use an analogous method to the one just described to obtain the existence theory of a variational problem, similar to the one governing steady 2-dimensional ideal fluid flows containing symmetric vortex pairs. Let Π be the half plane defined in $\mathbb{R}^2$ by $x_2 > 0$. Let $\zeta_0 \in L^p(\Pi)$ $(p > 2)$ be a non-negative function having support of finite measure. For any positive λ and any real positive number $n \geq 1$, we consider the variational problem

$$\max_{\zeta \in \mathcal{F}(\zeta_0)} (E(\zeta) - \lambda I_n(\zeta)), \tag{1.17}$$

where

$$I_n(\zeta) = \frac{1}{n} \int\limits_{\Pi} x_2^n \zeta(x) dx$$

and $E(\zeta)$ is defined as in (1.13). The main feature of our work is proving that for all $\lambda > 0$ and $n \geq 3$, the problem (1.17) has a solution. In other words, the functional $E(\zeta) - \lambda I_n(\zeta)$ attains a maximum value relative to $\mathcal{F}(\zeta_0)$ for all $\lambda > 0$ and $n \geq 3$. Also, we show that if $n = 2$, a case of physical interest, there exists $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$, the above functional has a maximiser relative to $\mathcal{F}(\zeta_0)$. In both cases, if ζ is a maximiser and $\psi := K\zeta$, then

$$-\Delta\psi = \phi \circ (\psi - \frac{\lambda}{n} x_2^n) \tag{1.18}$$

almost everywhere in Π, for some increasing function ϕ.

In Chapter 3, we employ a different variational formulation to study the same boundary-value problem. More precisely, for $n \geq 1$ an integer number, we study the variational problems

$$\max_{\zeta \in \mathcal{W}(\zeta_0), I_n(\zeta) = I} E(\zeta) \tag{1.19}$$

where ζ_0 satisfies similar assumptions to Chapter 2. This approach is an application of Burton's method [18] for vortex rings with prescribed impulse. We show equally that if ζ is a solution for (1.19) and $\psi := K\zeta$, then ψ satisfies (1.18) almost everywhere in Π for some increasing function ϕ and some unknown positive constant λ. We show also that if $n \geq 3$, then for any I, the solution $\zeta \in \mathcal{F}(\zeta_0)$. If $n \in \{1, 2\}$ and I is large enough, then also $\zeta \in \mathcal{F}(\zeta_0)$.

In Chapter 4, we prove an existence theorem for a variational problem related to vortex rings in three dimensions. For $(r, \theta, z) \in \mathbb{R}^3$, let Π be the half-plane defined by $r > 0$ and $-\infty < z < \infty$ and let $v_0 \in L^p(\Pi, \nu)$ be a non-negative function having support of finite measure with respect to $d\nu = 2\pi r dr dz$, where $p > 5/2$. For any positive λ and real number $n \geq 1$, we consider the variational problem

$$\max_{v \in \mathcal{F}(v_0)} (E(v) - \lambda I_{2n}(v)), \tag{1.20}$$

where

$$I_{2n}(v) = \frac{1}{2n} \int_{\Pi} r^{2n} v(r, z) d\nu,$$

$$E(v) = \frac{1}{2} \int_{\Pi} v(r, z) K v(r, z) d\nu$$

and K is the inverse of the operator $\mathcal{L}$ with homogeneous Dirichlet boundary conditions on Π. We show that if $n \geq 4$, then there exists a solution for (1.20) and if $\Psi := Kv$, where v is a solution for (1.20); then Ψ satisfies the equation

$$\mathcal{L}\Psi = \phi \circ (\Psi - \frac{\lambda}{2n} r^{2n})$$

almost everywhere for some increasing function ϕ. Also, if $n \in \{2, 3\}$ then, by making a certain unproved plausible assumption, we can prove the existence of a positive $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$, then (1.20) has a solution. We call the case when $n = 2$, the existence of a vortex ring in a Poiseuille flow.

Chapter 2

An existence theorem for a steady vortex in a planar domain

2.1 Introduction

In this Chapter, we study the existence theory of steady flows described by a variational problem similar to the one governing steady 2-dimensional ideal fluid flows containing symmetric vortex pairs. The flow in question is written in terms of a stream function $\psi : \Pi \to \mathbb{R}$ with ψ even in x_1, where Π is the half-plane defined in $\mathbb{R}^2$ by $x_2 > 0$. At infinity the stream function approaches $-\frac{\lambda}{n}x_2^n$ which representing a flow of velocity λx_2^{n-1}, in the negative x_1-direction, where $n \geq 1$. The vorticity is described by $-\Delta\psi$, where Δ is the Laplacian operator in two dimensions, $-\Delta\psi$ vanishes outside a bounded region placed symmetrically about the x_2 axis, and avoiding the x_1 axis. The vorticity in the region $x_2 > 0$ is non-negative, and ψ satisfies the equation

$$-\Delta\psi + \lambda(n-1)x_2^{n-2} = \phi(\psi), \tag{2.1}$$

where ϕ is an unknown function.

The results of this study prove that for $\lambda > 0$ and $n \geq 3$, a solution (ψ, ϕ) of this problem exists, where the vorticity field is a rearrangement of a prescribed non-negative function $\zeta_0 \in L^p$ $(p > 2)$ which has support of finite measure. If $n = 2$ and λ is sufficiently small, then a solution (ψ, ϕ) exists for which the vorticity field is a rearrangement of ζ_0. The existence theorem is based on maximising a functional related to the kinetic energy over the set of rearrangements of ζ_0 to obtain the vorticity ζ. This approach is an application of a theory proposed by Benjamin [5] for vortex rings in three dimensions. When $n = 1$ or $n = 2$, the equation (2.1) can be written as

$$-\Delta\psi = \phi(\psi), \tag{2.2}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ here is an increasing function. This equation for arbitrary ϕ is the equation for a stream function of a steady ideal fluid flow (see Lamb [30], page 244). Burton [13] was concerned with this problem, specifically in the case when $n = 1$. For small positive λ, he proved the existence of a solution to the equation (2.2) for which the vorticity is a rearrangement of a given non-negative function. In the case $n = 1$, other authors were concerned with similar problems. In particular, Badiani [3] proved the existence of steady planar flow of an ideal fluid past an obstacle, and Turkington [39, 40] showed the existence of vortex pairs in flows occupying the whole space $\mathbb{R}^2$ or $\mathbb{R}^2 \backslash D$, where D is a bounded, simply connected region, symmetric in the x_1-direction, containing the origin in its interior and having smooth boundary.

When $n = 1$, the solution can be extended by reflection from Π to $\mathbb{R}^2$ (making the stream function odd in x_2) to yield a symmetric vortex pair in an uniform flow. When $n = 2$ this reflection process yields a vortex pair in a "two phase" shear flow. The variational principle which will be used, has two mathematical difficulties. The first is in the nature of the set of rearrangements (as a subset of L^p). The second is in the loss of compactness which arises from the unbounded domain Π. In order to overcome these difficulties, the problem will be first solved on a bounded domain in Π by using Burton's results [10]. Therefore the functional for a bounded domain has a maximiser. In the second step, passing to the unbounded domain is accomplished by using some estimates to show that a solution in a sufficiently large bounded domain is in fact valid throughout the half-plane Π.

2.2 Mathematical formulation

For $\xi > 0$ and $X > 0$, we define the sets $\Pi(\xi, X) = \{x \in \Pi | |x_1| < \xi, x_2 < X\}$. We use $|A|$ to denote the 2-dimensional Lebesgue measure of a measurable set $A \subset \mathbb{R}^2$. The support of a function $\zeta : \Pi \to \mathbb{R}$ is $\operatorname{supp} \zeta = \{x \in \Pi | \zeta(x) \neq 0\}$. If there exists a positive constant C such that for all $x \in \operatorname{supp} \zeta$, $|x| \leq C$, then $\operatorname{supp} \zeta$ is bounded. For x, $y \in \mathbb{R}^2$ we set

$$G(x,y) = \frac{1}{2\pi} \log \left(\frac{|x - \bar{y}|}{|x - y|} \right) = \frac{1}{2\pi} \log \left(\frac{|\bar{x} - y|}{|x - y|} \right),$$

where $\bar{x} = (x_1, -x_2)$ and $\bar{y} = (y_1, -y_2)$; then G is the Green's function for $-\Delta$ with homogeneous Dirichlet boundary conditions on Π. If $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ for some $p > 1$, then for all x in $\mathbb{R}^2$, we define $K\zeta(x)$ by the absolutely convergent integral

$$K\zeta(x) := \int_{\Pi} G(x,y)\zeta(y)dy.$$

Moreover $K\zeta$ is the weak solution in the distribution sense to the problem $-\Delta\psi = \zeta$ in Π, $\psi(x_1, 0) = 0$ and $\psi(x) \to 0$ as $|x| \to \infty$. For all $\zeta \geq 0$, λ and n we set

$$\Phi_\lambda^n(\zeta) = E(\zeta) - \lambda I_n(\zeta),$$

where $E(\zeta)$ and $I_n(\zeta)$ are as defined in Chapter 1.

Now with all this notation, our main result is as follows.

Theorem 2.1. *Let $n \geq 2$, let $2 < p < \infty$ and let $\zeta_0 \in L^p(\Pi)$ be a non-negative function having support of finite measure. Let $\mathcal{F}(\zeta_0)$ be the set of rearrangements of ζ_0 on Π.*

- *(i) If $n \geq 3$, then for any $\lambda > 0$, the functional Φ_λ^n attains a maximum value relative to $\mathcal{F}(\zeta_0)$.*
- *(ii) If $n = 2$, then there exists a positive constant Λ such that the functional Φ_λ^2 attains a maximum value relative to $\mathcal{F}(\zeta_0)$ for each fixed $\lambda \in (0, \Lambda)$.*

In both cases if ζ is a maximiser and $\psi := K\zeta$, then we have

$$-\Delta\psi = \phi \circ (\psi - \frac{\lambda}{n} x_2^n)$$

almost everywhere in Π for some increasing function $\phi : \mathbb{R} \to \mathbb{R}$.

Later, we will see that the key for proving this Theorem depends on showing that if ζ is a maximiser of Φ_λ^n relative $\mathcal{J}(\xi, X)$, the set of all rearrangements of ζ_0 on $\Pi(\xi, X)$ for various ξ and X depending on λ, then the measure of the set where $K\zeta(x) - \frac{\lambda}{n} x_2^n$ is positive has a measure greater than $|\text{supp}\ \zeta_0|$. The case when $n = 2$ can be proved by using two methods. The first is similar to the one which has been used by Burton [13]; the second is based on proving that the measure of the set where $K\zeta(x) - \frac{\lambda}{2} x_2^2 > 0$ tends to ∞ when $\lambda \to 0$. The second method will be used as a starting point to show that, if a certain assumption could be proved, then the Conjecture of Chapter 4 would become a Theorem.

2.3 Estimates and properties for the function $K\zeta$

We begin our study with some results concerning the properties of the operator K, which allow us to derive some estimates for $K\zeta$ used throughout this Chapter.

Lemma 2.2. *For any x, y in Π and real $k \geq 1$, we have*

$$\frac{x_2 y_2}{\pi |x - \bar{y}|^2} \leq G(x, y) \leq \frac{2^{1/k} k (x_2 y_2)^{1/2k}}{2\pi |x - y|^{1/k}}.$$

Proof. For $t \geq 1$, we have

$$\frac{1}{t^2} \leq \frac{1}{t} \leq \frac{1}{\sqrt{t}}, \tag{2.3}$$

and also for x, y in Π

$$|x - \bar{y}| \geq |x - y|.$$

Then from (2.3) we have

$$\log\left(\frac{|x - \bar{y}|}{|x - y|}\right) = \int_1^{\frac{|x-\bar{y}|}{|x-y|}} \frac{dt}{t} \geq \int_1^{\frac{|x-\bar{y}|}{|x-y|}} \frac{dt}{t^2} = 1 - \frac{|x - y|}{|x - \bar{y}|}.$$

Also from (2.3) and for all $k \in [1, \infty)$ we can write

$$\log\left(\frac{|x-\bar{y}|}{|x-y|}\right) = \frac{k}{2}\log\left(\frac{|x-\bar{y}|}{|x-y|}\right)^{2/k} \leq \frac{k}{2}\int_1^{(\frac{|x-\bar{y}|}{|x-y|})^{2/k}} \frac{dt}{\sqrt{t}} = k\left(\left(\frac{|x-\bar{y}|}{|x-y|}\right)^{1/k} - 1\right).$$

We now set

$$G_1(x,y) = 1 - \frac{|x-y|}{|x-\bar{y}|} \quad \text{and} \quad G_2(x,y) = \left(\frac{|x-\bar{y}|}{|x-y|}\right)^{1/k} - 1.$$

For G_1, we have

$$G_1(x,y) = \frac{|x-\bar{y}| - |x-y|}{|x-\bar{y}|} = \frac{|x-\bar{y}|^2 - |x-y|^2}{(|x-\bar{y}| + |x-y|)|x-\bar{y}|} \geq \frac{2x_2y_2}{|x-\bar{y}|^2}. \tag{2.4}$$

Since

$$|x-\bar{y}|^2 = |x-y|^2 + 4(x_2y_2);$$

then for all x and y in Π we have

$$|x-\bar{y}| \leq |x-y| + 2(x_2y_2)^{1/2}.$$

Hence for all $k \geq 1$ we get

$$|x-\bar{y}|^{1/k} \leq |x-y|^{1/k} + 2^{1/k}(x_2y_2)^{1/2k}.$$

Thus we find

$$G_2(x,y) \leq \frac{2^{1/k}(x_2y_2)^{1/2k}}{|x-y|^{1/k}}. \tag{2.5}$$

Therefore the Lemma follows from (2.4) and (2.5). □

The following estimate is not optimal, but it is sufficient for use an alternative proof of Theorem 2.1 in the case when $n = 2$.

Lemma 2.3. *For any non-negative function $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ $(p > 1)$, and for all $x \in \Pi$ we have*

$$K\zeta(x) \geq \frac{x_2}{2\pi(1+|x|^2)} \int_\Pi \frac{y_2}{1+|y|^2}\zeta(y)dy.$$

Furthermore

$$E(\zeta) \geq \frac{1}{4\pi}\left(\int_\Pi \frac{y_2}{1+|y|^2}\zeta(y)dy\right)^2.$$

Proof. Since ζ is non-negative, then by using Lemma 2.3 we have

$$K\zeta(x) \geq \frac{1}{\pi}\int_\Pi \frac{x_2y_2}{|x-\bar{y}|^2}\zeta(y)dy.$$

On the other hand, for all x and y in Π we have

$$\begin{aligned}
|x-\bar{y}|^2 &\leq (|x|+|y|)^2 \\
&\leq 2|x|^2+2|y|^2 \\
&< 2(1+|x|^2)(1+|y|^2).
\end{aligned}$$

Hence we find

$$K\zeta(x) \geq \frac{x_2}{2\pi(1+|x|^2)} \int\limits_{\Pi} \frac{y_2}{1+|y|^2}\zeta(y)dy. \tag{2.6}$$

Applying the definition of E and (2.6) we find for $\zeta \geq 0$

$$E(\zeta) \geq \frac{1}{4\pi} \int\limits_{\Pi} \frac{x_2}{1+|x|^2}\zeta(x)dx \int\limits_{\Pi} \frac{y_2}{1+|y|^2}\zeta(y)dy = \frac{1}{4\pi}\left(\int\limits_{\Pi} \frac{y_2}{1+|y|^2}\zeta(y)dy\right)^2.$$

This completes the proof. □

Lemma 2.4. *Let $k \geq 1$ and let $\frac{2k}{2k-1} < p < \infty$. Let $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ be a non-negative function. Then for all $x \in \Pi$, we have*

$$K\zeta(x) \leq C(\|\zeta\|_1 + \|\zeta\|_p)(x_2^{1/2k} + x_2^{1/k}),$$

where C is a positive constant that depends only on p and k.

Proof. For all x and y in Π, we have

$$x_2y_2 = x_2(y_2-x_2) + x_2^2 \leq x_2|x-y| + x_2^2,$$

then for all $k \geq 1$

$$(x_2y_2)^{1/2k} \leq x_2^{1/2k}|x-y|^{1/2k} + x_2^{1/k}. \tag{2.7}$$

Now from Lemma 2.2 and (2.7), we have

$$\begin{aligned}
K\zeta(x) &\leq \frac{2^{1/k}k}{2\pi} \int\limits_{\Pi} \frac{x_2^{1/2k}|x-y|^{1/2k} + x_2^{1/k}}{|x-y|^{1/k}}\zeta(y)dy \\
&= \frac{2^{1/k}k}{2\pi} \int\limits_{\Pi} \frac{x_2^{1/2k}}{|x-y|^{1/2k}}\zeta(y)dy + \frac{2^{1/k}k}{2\pi} \int\limits_{\Pi} \frac{x_2^{1/k}}{|x-y|^{1/k}}\zeta(y)dy. \qquad (2.8)
\end{aligned}$$

We set

$$F_1(x) = \int\limits_{\Pi} \frac{x_2^{1/2k}}{|x-y|^{1/2k}}\zeta(y)dy \quad \text{and} \quad F_2(x) = \int\limits_{\Pi} \frac{x_2^{1/k}}{|x-y|^{1/k}}\zeta(y)dy.$$

So we can write $F_1(x)$ as

$$F_1(x) = \left(\int\limits_{|x-y|\le 1} + \int\limits_{|x-y|>1} \right) \frac{x_2^{1/2k}}{|x-y|^{1/2k}} \zeta(y)dy.$$

Then by using Hölder's inequality, we get

$$\begin{aligned} F_1(x) &\le x_2^{1/2k} \|\zeta\|_p \left(\int\limits_{|x-y|\le 1} \frac{dy}{|x-y|^{q/2k}} \right)^{1/q} + x_2^{1/2k} \|\zeta\|_1 \\ &= x_2^{1/2k} \|\zeta\|_p (2\pi \int_0^1 \frac{r}{r^{q/2k}} dr)^{1/q} + x_2^{1/2k} \|\zeta\|_1 \\ &= (\frac{4k\pi}{4k-q})^{1/q} \|\zeta\|_p x_2^{1/2k} + \|\zeta\|_1 x_2^{1/2k}, \end{aligned} \tag{2.9}$$

where q is the conjugate exponent of p, so $q < 2k$. Performing a similar calculation for $F_2(x)$ we find that

$$F_2(x) \le ((\frac{2k\pi}{2k-q})^{1/q} \|\zeta\|_p x_2^{1/k} + \|\zeta\|_1 x_2^{1/k}. \tag{2.10}$$

Thus from (2.8), (2.9) and (2.10) we have

$$K\zeta(x) \le C(\|\zeta\|_1 + \|\zeta\|_p)(x_2^{1/2k} + x_2^{1/k}),$$

where $C = \frac{2^{1/k}k}{2\pi}((\frac{2k\pi}{2k-q})^{1/q} + 1)$. □

Lemma 2.5. *Let $k \ge 1$ and let $\frac{2k}{2k-1} < p < \infty$. Then there is a positive number M such that*

$$K\zeta(x) \le M(\|\zeta\|_1 + \|\zeta\|_p)(x_2^{1/2k} + x_2^{1/k}) \min\{1, |x_1|^{\frac{-1}{2k+p}}\}$$

for all $x \in \Pi$, whenever $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ is Steiner-symmetric.

Proof. The method of proof which will be used here is the same as [13, Lemma 5]. Recall that if $u \in L^p(\Pi)$ is Steiner symmetric and if $|x_1| \ge b$ then

$$\int\limits_{|x_1-y_1|<b} u(y)dy \le \frac{b}{|x_1|} \int\limits_{\Pi} u(y)dy. \tag{2.11}$$

Suppose $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ is Steiner symmetric, let $x \in \Pi$ be fixed and for $y \in \Pi$, let ζ_1 be defined by

$$\zeta_1(y) = \begin{cases} \zeta(y) & \text{if } |x_1 - y_1| < |x_1|^{\frac{2k}{2k+p}}, \\ 0 & \text{if } |x_1 - y_1| \ge |x_1|^{\frac{2k}{2k+p}}. \end{cases}$$

Then by using Lemma 2.4 and (2.11), we get

$$K\zeta_1(x) \le C(x_2^{1/2k} + x_2^{1/k})(\|\zeta_1\|_1 + \|\zeta_1\|_p)$$

$$
\begin{aligned}
&\le C(x_2^{1/2k}+x_2^{1/k})\Big(\frac{|x_1|^{\frac{2k}{2k+p}}}{|x_1|}\|\zeta\|_1+\Big(\frac{|x_1|^{\frac{2k}{2k+p}}}{|x_1|}\Big)^{1/p}\|\zeta\|_p\Big)\\
&= C(x_2^{1/2k}+x_2^{1/k})(|x_1|^{\frac{-p}{2k+p}}\|\zeta\|_1+|x_1|^{\frac{-1}{2k+p}}\|\zeta\|_p), \qquad (2.12)
\end{aligned}
$$

where C is a positive constant defined as in Lemma 2.4. We now set $\zeta_2=\zeta-\zeta_1$, then by using Lemma 2.2 and (2.7), we get

$$
\begin{aligned}
K\zeta_2(x) &= K\zeta(x)-K\zeta_1(x)\\
&\le \frac{1}{2\pi}\int\limits_{|x_1-y_1|\ge|x_1|^{\frac{2k}{2k+p}}} \log\left(\frac{|x-\bar{y}|}{|x-y|}\right)\zeta(y)dy\\
&\le \frac{2^{1/k}k}{2\pi}\int\limits_{|x_1-y_1|\ge|x_1|^{\frac{2k}{2k+p}}} \frac{(x_2y_2)^{1/2k}}{|x-y|^{1/k}}\zeta(y)dy\\
&\le \frac{2^{1/k}k}{2\pi}\int\limits_{|x_1-y_1|\ge|x_1|^{\frac{2k}{2k+p}}} \frac{x_2^{1/2k}|x-y|^{1/2k}+x_2^{1/k}}{|x-y|^{1/k}}\zeta(y)dy\\
&\le \frac{2^{1/k}k}{2\pi}\int\limits_{|x_1-y_1|\ge|x_1|^{\frac{2k}{2k+p}}} \frac{x_2^{1/2k}}{|x-y|^{1/2k}}\zeta(y)dy\\
&\quad+\frac{2^{1/k}k}{2\pi}\int\limits_{|x_1-y_1|\ge|x_1|^{\frac{2k}{2k+p}}} \frac{x_2^{1/k}}{|x-y|^{1/k}}\zeta(y)dy\\
&\le \frac{2^{1/k}k}{2\pi}\left(x_2^{1/2k}|x_1|^{\frac{-1}{2k+p}}+x_2^{1/k}|x_1|^{\frac{-2}{2k+p}}\right)\|\zeta\|_1\\
&\le \frac{2^{1/k}k}{2\pi}(x_2^{1/2k}+x_2^{1/k})(|x_1|^{\frac{-1}{2k+p}}+|x_1|^{\frac{-2}{2k+p}})\|\zeta\|_1. \qquad (2.13)
\end{aligned}
$$

Thus from (2.12) and (2.13) the required inequality follows. □

Lemma 2.6. *Let $2<p\le\infty$ and let $\zeta\in L^1(\Pi)\cap L^p(\Pi)$. Then there there exists a positive constant N such that for all $x\in\Pi$*

$$|\nabla K\zeta(x)|\le N(\|\zeta\|_1+\|\zeta\|_p).$$

Proof. From [13, Lemma 2], we have

$$|\nabla_x G(x,y)|\le\frac{1}{\pi|x-y|}$$

for all $x\in\Pi$ and $y\in\Pi$. Then it follows that

$$|\nabla K\zeta(x)|\le\frac{1}{\pi}\int\limits_{\Pi}\frac{|\zeta(y)|}{|x-y|}dy.$$

Therefore we find

$$\begin{aligned}|\nabla K\zeta(x)| &\leq \frac{1}{\pi}\int\limits_{|x-y|<1}\frac{|\zeta(y)|}{|x-y|}dy+\frac{1}{\pi}\int\limits_{|x-y|\geq 1}\frac{|\zeta(y)|}{|x-y|}dy\\ &\leq \frac{1}{\pi}\int\limits_{|x-y|<1}\frac{|\zeta(y)|}{|x-y|}dy+\frac{1}{\pi}\|\zeta\|_1.\end{aligned}$$

If $2<p<\infty$, then by using Höder inequality

$$|\nabla K\zeta(x)|\leq \frac{1}{\pi}(\frac{2\pi}{2-q})^{1/q}\|\zeta\|_p+\frac{1}{\pi}\|\zeta\|_1.$$

If $p=\infty$, then it follows that

$$|\nabla K\zeta(x)|\leq 2\|\zeta\|_\infty+\frac{1}{\pi}\|\zeta\|_1.$$

Hence the result follows. □

We end this section by stating some results from [13], which will be used in the next sections.

Lemma 2.7. *Let $2\leq p<\infty$, let q be the conjugate exponent of p and let Ω be a bounded open subset of Π. Then $K: L^p(\Omega)\to L^q(\Omega)$ is compact in the sense that if $\{\zeta_n\}_{n=1}^\infty$ is a sequence of functions bounded in $L^p(\Pi)$ and vanishing outside Ω, then the restriction to Ω of $K\zeta_n$ has a subsequence converging in the q-norm .*

Proof. The case when $p>2$ has been proved in Burton [13, Lemma 8], so we need just to prove the case where $p=2$. From Lemma 2.2 we have

$$G(x,y)\leq\frac{2^{1/k}k(x_2y_2)^{1/2k}}{2\pi|x-y|^{1/k}}$$

for all $k\geq 1$ and all x and y in Π, so

$$\int\limits_\Omega\int\limits_\Omega|G(x,y)|^2dxdy\leq\frac{2^{2/k}k^2}{4\pi^2}\int\limits_\Omega\int\limits_\Omega\frac{(x_2y_2)^{1/k}}{|x-y|^{2/k}}dxdy<\infty,$$

hence $G\in L^2(\Omega\times\Omega)$. Since $L^2(\Omega)$ is a Hilbert space, then it follows from [7, Theorem vi] that K is Hilbert-Schmidt operator, so K is compact. □

Lemma 2.8. *Let $2<p<\infty$ and let $\zeta\in L^p(\Pi)$ have bounded support. Then $\nabla K\zeta(x)=O(|x|^{-2})$ and $K\zeta(x)=O(|x|^{-1})$ as $|x|\to\infty$, and*

$$\int\limits_\Pi|\nabla K\zeta|^2dx=\int\limits_\Pi\zeta K\zeta dx<\infty.$$

This Lemma shows that the operator K is strictly positive on $L^p(\Omega)$ for any open subset Ω in Π and for $p > 2$.

Lemma 2.9. *Let $\zeta \in L^1(\Pi)$ be a non-negative function and have bounded support. Then*

(i) $\int\limits_{\Pi} \zeta(x) K\zeta(x)dx \leq \int\limits_{\Pi} \zeta^s(x) K\zeta^s(x)dx$,

(ii) if ζ is Steiner-symmetric then $K\zeta$ is Steiner -symmetric.

Lemma 2.10. *Let $1 < p < \infty$, let $0 < a < \infty$, let $\zeta_0 \in L^p(\Pi)$ and suppose that $\zeta_0(x)$ is a nontrivial decreasing function of x that vanishes for $|x| \geq a$. For $t \geq a$ define $\zeta_t(x_1, x_2) = \zeta_0(x_1, x_2 - t)$. Then there exists a positive constant C such that*

$$\int\limits_{\Pi} \zeta_t(x) K\zeta_t(x)dx \geq C \log t$$

for sufficiently large t.

2.4 Properties of the functional Φ_λ^n

In this section, estimates for the function $K\zeta$ found in section 3 will be used to give some properties for the functional Φ_λ^n. Throughout this section and the following section we shall use 1_Ω to denote the characteristic function of a measurable set $\Omega \subset \Pi$ defined by $1_\Omega(x) = 1$ if $x \in \Omega$ and $1_\Omega(x) = 0$ if $x \notin \Omega$.

Lemma 2.11. *Let $n \geq 1$, let $\lambda > 0$, let $k \geq 1$, let $\frac{2k}{2k-1} < p < \infty$ and let $\zeta_0 \in L^p(\Pi)$ be a non-negative function having support of finite measure. Let $\mathcal{R}(\zeta_0)$ be the set of rearrangments of ζ_0 on Π having bounded support. Then there exists a positive number X depending on λ such that if $\zeta \in \mathcal{R}(\zeta_0)$ and does not vanish almost everywhere for $x_2 > X$, we can choose a positive number ξ for which we have*

(a) $\Phi_\lambda^n(\zeta_1) > \Phi_\lambda^n(\zeta)$,

(b) $\Phi_\lambda^n(\zeta_2) > \Phi_\lambda^n(\zeta)$,

where $\zeta_1 = \zeta 1_{\{x_2 < X\}}$ and $\zeta_2 \in \mathcal{R}(\zeta_0)$ is some function such that $\operatorname{supp} \zeta_2 \subset \Pi(\xi, X)$.

Proof. From Lemma 2.4, we have

$$K\zeta(x) \leq C\|\zeta\|_p x_2^{1/k}$$

for $x_2 > 1$; then

$$K\zeta(x) - \frac{\lambda}{n} x_2^n \leq C\|\zeta\|_p x_2^{1/k} - \frac{\lambda}{n} x_2^n,$$

so we can take $X(\lambda) = (\frac{nC\|\zeta\|}{\lambda})^{\frac{k}{nk-1}}$, to ensure that for $x_2 > X = \max\{1, X(\lambda)\}$

$$K\zeta(x) - \frac{\lambda}{n} x_2^n < 0.$$

Now for $\zeta \in L^p(\Pi)$ we assume that $\zeta_1 = \zeta 1_{\{x_2<X\}} \neq 0$ and set $\bar{\zeta} = \zeta 1_{\{x_2 \geq X\}}$. By using Lemma 2.8, K is strictly positive, then the functional Φ_λ^n is strictly convex. Hence it follows

$$\begin{aligned} \Phi_\lambda^n(\zeta_1) &= \Phi_\lambda^n(\zeta - \bar{\zeta}) \\ &> \Phi_\lambda^n(\zeta) + \int\limits_\Pi (-K\zeta(x) + \frac{\lambda}{n} x_2^n)\bar{\zeta}(x)dx. \end{aligned} \tag{2.14}$$

Thus

$$\Phi_\lambda^n(\zeta_1) > \Phi_\lambda^n(\zeta).$$

To prove (b), we assume that ζ does not vanish almost everywhere for $x_2 > X$. Then the function defined by $\bar{\zeta} = \zeta 1_{\{x_2 \geq X\}}$ is not almost everywhere zero. Hence if we choose $\xi > 0$ and $0 < \varepsilon < X$ such that $2\xi\varepsilon > \pi a^2$, where $|\text{supp } \zeta_0| = \pi a^2$, then we can construct a rearrangement $\widetilde{\zeta}$ of $\bar{\zeta}$ that satisfies

$$\text{supp } \widetilde{\zeta} \subset (-\xi, \xi) \times (0, \varepsilon) \quad \text{and} \quad \text{supp } \widetilde{\zeta} \cap \text{supp } \zeta_1 = \emptyset.$$

Indeed, let $\xi > 0$ and $\varepsilon < X$ be as above; then there is a measurable set A that satisfies $A \subset (-\xi, \xi) \times (0, \varepsilon)$, $A \cap \text{supp } \zeta_1 = \emptyset$ and $|A| = |\text{supp } \bar{\zeta}|$, so by using the Theorem of isomorphism of measure spaces [27], there is an isomorphism $\sigma : A \to \text{supp } \bar{\zeta}$ such that $|A| = |\text{supp } \bar{\zeta}|$. Hence by setting $\widetilde{\zeta} = \bar{\zeta} \circ \sigma$, we find that $\widetilde{\zeta}$ is a rearrangement of $\bar{\zeta}$ supported by the measurable set A. Thus it follows that

$$\text{supp } \widetilde{\zeta} \cap \text{supp } \zeta_1 = \emptyset,$$

and therefore the function $\zeta_1 + \widetilde{\zeta}$ is a rearrangement of ζ. We need now just to show that

$$\Phi_\lambda^n(\zeta_1 + \widetilde{\zeta}) - \Phi_\lambda^n(\zeta_1) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

For that we have

$$\begin{aligned} \Phi_\lambda^n(\zeta_1 + \widetilde{\zeta}) &= \Phi_\lambda^n(\zeta_1) + \frac{1}{2}\int\limits_\Pi \zeta_1(x)K\widetilde{\zeta}(x)dx + \frac{1}{2}\int\limits_\Pi \widetilde{\zeta}(x)K\zeta_1(x)dx \\ &\quad + \frac{1}{2}\int\limits_\Pi \widetilde{\zeta}(x)K\widetilde{\zeta}(x)dx - \frac{\lambda}{n}\int\limits_\Pi x_2^n \widetilde{\zeta}(x)dx. \end{aligned}$$

Since G is symmetric, then it follows that K is symmetric. Thus we have

$$\begin{aligned} \Phi_\lambda^n(\zeta_1 + \widetilde{\zeta}) - \Phi_\lambda^n(\zeta_1) &= \int\limits_\Pi \zeta_1(x)K\widetilde{\zeta}(x)dx + \frac{1}{2}\int\limits_\Pi \widetilde{\zeta}(x)K\widetilde{\zeta}(x)dx - \frac{\lambda}{n}\int\limits_\Pi x_2^n\widetilde{\zeta}(x)dx \\ &< \int\limits_\Pi \zeta_1(x)K\widetilde{\zeta}(x)dx + \frac{1}{2}\int_\Pi \widetilde{\zeta}(x)K\widetilde{\zeta}(x)dx. \end{aligned}$$

On the other hand we have

$$\Phi_\lambda^n(\zeta_1+\widetilde{\zeta})-\Phi_\lambda(\zeta_1)\geq -\frac{\lambda}{n}\int\limits_{\Pi} x_2^n\widetilde{\zeta}(x)dx,$$

then

$$\Phi_\lambda^n(\zeta_1+\widetilde{\zeta})-\Phi_\lambda(\zeta_1)\geq -\frac{\lambda}{n}\varepsilon^n\|\widetilde{\zeta}\|_1.$$

By using Lemma 2.4 and the above estimate, we find that

$$-\frac{\lambda\varepsilon^n}{n}\|\widetilde{\zeta}\|_1\leq \Phi_\lambda^n(\zeta_1+\widetilde{\zeta})-\Phi_\lambda^n(\zeta_1)\leq C\|\zeta_0\|_p(\varepsilon^{1/2k}+\varepsilon^{1/k}).$$

Therefore

$$\Phi_\lambda^n(\zeta_1+\widetilde{\zeta})-\Phi_\lambda^n(\zeta_1)\to 0 \quad \text{as } \varepsilon\to 0.$$

Thus, for small $\varepsilon>0$

$$\Phi_\lambda^n(\zeta_1+\widetilde{\zeta})>\Phi_\lambda^n(\zeta).$$

This completes the proof. □

Remark 2.12. *In order to prove the main Theorem 2.1 in section 2.5, we use Lemma 2.11 to replace a maximising sequence for Φ_λ^n relative to set $\mathcal{R}(\zeta_0)$, by a sequence of maximisers of Φ_λ^n relative to the sets*

$$\mathcal{J}(\xi,X)=\{\zeta\in\mathcal{R}(\zeta_0)|\operatorname{supp}\zeta\subset\Pi(\xi,X)\},$$

for various ξ, which again give a maximising sequence relative to $\mathcal{R}(\zeta_0)$.

The following Lemma concerns the relationship between the supremum of Φ_λ^n over the set $\mathcal{R}(\zeta_0)$ as defined in Lemma 2.11, and the supremum of Φ_λ^n over $\mathcal{F}(\zeta_0)$. Note that in general, the set $\mathcal{F}(\zeta_0)$ is likely to have some functions with unbounded support, which are rearrangements of ζ_0.

Lemma 2.13. *Let the assumption about n, λ, k, p, ζ_0 and $\mathcal{R}(\zeta_0)$ be the same as in Lemma 2.11. Then we have*

$$\sup_{\mathcal{F}(\zeta_0)}\Phi_\lambda^n=\sup_{\mathcal{R}(\zeta_0)}\Phi_\lambda^n.$$

Proof. Let X be defined as in Lemma 2.11. There are two steps in the proof of this Lemma. The first step is to show that for given ζ in $\mathcal{F}(\zeta_0)$ with unbounded support, and if $\bar{\zeta}=\zeta 1_{\mathbb{R}\times(0,X)}$ then we have

$$\Phi_\lambda^n(\bar{\zeta})\geq\Phi_\lambda^n(\zeta).$$

The second step is to show that for $\delta>0$ arbitrary, there exists a rearrangement ζ^* of $\zeta-\widetilde{\zeta}$ supported by $\Pi(Y,X)\backslash\operatorname{supp}\zeta$ for some Y such that

$$\Phi_\lambda^n(\zeta^*+\widetilde{\zeta})\geq\Phi_\lambda^n(\zeta)-\delta,$$

where $\widetilde{\zeta} = \zeta 1_{\Pi(Y,X)}$.

Recalling that $\bar{\zeta} = \zeta 1_{\mathbb{R}\times(0,X)}$, then by using the same method as in Lemma 2.11 (a), we have

$$\Phi_\lambda^n(\bar{\zeta}) \geq \Phi_\lambda^n(\zeta).$$

We are going now to construct ζ^*. Indeed, let $0 < a < \infty$ be such that $|\text{supp } \zeta_0| = \pi a^2$, let $\varepsilon < X$ and Y be two positive numbers chosen so that $2\varepsilon Y > \pi a^2$; then there exists a measurable set B that satisfies $B \subset \Pi(\varepsilon, Y)$, $B \cap \text{supp } (\zeta 1_{\Pi(\varepsilon,Y)}) = \emptyset$ and $|B| = \pi a^2 - |\text{supp } \zeta \cap \Pi(\varepsilon, Y)|$. Therefore by applying the Theorem of isomorphism of measure spaces [27], there exists an isomorphism $\sigma : \text{supp } (\zeta 1_{\Pi\setminus\Pi(\varepsilon,Y)}) \to B$; hence by setting $\zeta^* = \zeta \circ \sigma^{-1}$, we find that ζ^* is a rearrangement of $\zeta - \widetilde{\zeta}$, where $\widetilde{\zeta} = \zeta 1_{\Pi(Y,X)}$, supported by the measurable set B. It remains just to prove that for $\delta > 0$ arbitrary, and by suitable choice of ε and Y we can ensure that

$$\Phi_\lambda^n(\zeta^* + \widetilde{\zeta}) \geq \Phi_\lambda^n(\zeta) - \delta.$$

We may write $\bar{\zeta}$ as $\bar{\zeta} = \widetilde{\zeta} + \zeta_*$, where $\zeta_* = \zeta 1_{\mathbb{R}\times(0,X)\setminus\Pi(Y,X)}$. Then by using the properties of the operator K we have

$$\begin{aligned}
\Phi_\lambda^n(\bar{\zeta}) &= \Phi_\lambda^n(\widetilde{\zeta} + \zeta_*) \\
&= \Phi_\lambda^n(\widetilde{\zeta}) + \Phi_\lambda^n(\zeta_*) + \int_\Pi \zeta_*(x) K\widetilde{\zeta}(x)dx. \qquad (2.15)
\end{aligned}$$

Also we have

$$\Phi_\lambda^n(\widetilde{\zeta} + \zeta^*) = \Phi_\lambda^n(\widetilde{\zeta}) + \Phi_\lambda^n(\zeta^*) + \int_\Pi \zeta^*(x) K\widetilde{\zeta}(x)dx.$$

Since $\Phi_\lambda^n(\zeta_*) < \infty$, then it follows from Lemma 2.4 and (2.15) that

$$\begin{aligned}
\Phi_\lambda^n(\bar{\zeta}) - \Phi_\lambda^n(\widetilde{\zeta} + \zeta^*) &= \Phi_\lambda^n(\zeta_*) - \Phi_\lambda^n(\zeta^*) + \int_\Pi (\zeta_*(x) - \zeta^*(x)) K\widetilde{\zeta}(x)dx \\
&\leq \frac{1}{2}\int_\Pi \zeta_*(x) K\zeta_*(x)dx + \frac{\lambda}{n}\int_\Pi x_2^n \zeta^*(x)dx + \int_\Pi \zeta_*(x) K\widetilde{\zeta}(x)dx \\
&\leq \frac{3C}{2}\|\zeta_0\|_p \int_\Pi (x_2^{1/2k} + x_2^{1/k})\zeta_*(x)dx + \frac{\lambda}{n}\varepsilon^n \|\zeta\|_1.
\end{aligned}$$

Since $\|\zeta_*\|_1 \to 0$ as $Y \to \infty$, then by using Lebesgue's Dominated Convergence Theorem [27. Theorem D] we find that

$$\int_\Pi (x_2^{1/2k} + x_2^{1/k})\zeta_*(x)dx \to 0 \quad \text{as} \quad \zeta \to \infty.$$

Hence we can find $Y_0 > 0$ such that for all $Y > Y_0$ we have

$$\int_{\Pi} (x_2^{1/2k} + x_2^{1/k})\zeta_*(x)dx \leq \varepsilon^n.$$

Therefore we get

$$\Phi_\lambda^n(\bar{\zeta}) - \Phi_\lambda^n(\widetilde{\zeta} + \zeta^*) \leq (\frac{3C}{2} + \frac{\lambda}{n}\|\zeta_0\|_1)\varepsilon^n.$$

Now for $\delta > 0$ arbitrary we set

$$\varepsilon_0 = (\frac{2n\delta}{3nC + 2\lambda\|\zeta_0\|_1})^{1/n},$$

then by choosing $\varepsilon \leq \varepsilon_0$ we have

$$\Phi_\lambda^n(\bar{\zeta}) - \Phi_\lambda^n(\widetilde{\zeta} + \zeta^*) \leq \delta$$

for all $Y \geq Y_0$ with $2\varepsilon Y \geq \pi a^2$. Therefore by using the first step we find that for $\delta > 0$ arbitrary, we can choose a rearrangement $\zeta^* + \widetilde{\zeta}$ of ζ having bounded support such that

$$\Phi_\lambda^n(\zeta^* + \widetilde{\zeta}) \geq \Phi_\lambda^n(\zeta) - \delta.$$

This completes the proof. □

The next Lemmas in this section have the same ideas as in [17, Lemma 1 and 3].

Lemma 2.14. *Let the assumptions about λ, n, k, p and ζ_0 be the same as in Lemma 2.11. Let $t > 0$ be small enough that $|\zeta_0^{-1}(t,\infty)| > 0$. Let $N \in \mathbb{R}$ be such that $N^2 = |\zeta_0^{-1}(t,\infty)|$. Then a rearrangement ζ of ζ_0 can be chosen such that $\zeta(x) \geq t1_{[0,N]^2}(x)$. Define ζ_ρ by*

$$\zeta_\rho(x) = \zeta(\rho x_1, \rho^{-1}x_2)$$

for all $x \in \Pi$, where $\rho \in (0,1]$. Then for every $\lambda > 0$ and $n \geq 3$ we have

$$\Phi_\lambda^n(\zeta_\rho) > 0$$

for all sufficiently small positive ρ.

Proof. Let $l > 0$ be such that $|\text{supp } \zeta_0| = l^2$. For small enough $t > 0$, we set

$$S_t = \{x \in \Pi | \zeta_0(x) > t\},$$

so we have $S_t \subset \text{supp } \zeta_0$. Hence by using the Theorem of isomorphism of measure spaces [27], there exists an isomorphism $\sigma : [0,N]^2 \to S_t$. Now we let τ denote a measure-preserving bijection defined from $[0,l]^2\backslash[0,N]^2$ into supp $\zeta_0\backslash S_t$, and also let ζ be a function defined by

$$\zeta(x) = \begin{cases} \zeta_0 \circ \sigma(x) & \text{if } x \in [0,N]^2, \\ \zeta_0 \circ \tau(x) & \text{if } x \in [0,l]^2\backslash[0,N]^2. \end{cases}$$

Then ζ is a non-negative function and also is a rearrangement of ζ_0. Further $\zeta(x) \geq t1_{[0,N]^2}(x)$. It remain just to prove that if $n \geq 3$ and $\lambda > 0$, then for small $\rho > 0$,

$$\Phi_\lambda^n(\zeta_\rho) > 0.$$

Indeed, for ζ_ρ defined as above we have

$$\Phi_\lambda^n(\zeta_\rho) = E(\zeta_\rho) - \lambda I_n(\zeta_\rho).$$

Then by making linear change of variable, we find that

$$I_n(\zeta_\rho) = \rho^n I_n(\zeta). \tag{2.16}$$

Now by using Lemma 2.2, we have

$$\begin{aligned}
E(\zeta_\rho) &= \frac{1}{2\pi}\int\limits_\Pi\int\limits_\Pi \log\left(\frac{|x-\bar{y}|}{|x-y|}\right)\zeta_\rho(x)\zeta_\rho(y)dxdy \\
&\geq \frac{1}{\pi}\int\limits_\Pi\int\limits_\Pi \frac{x_2y_2}{|x-\bar{y}|^2}\zeta(\rho x_1,\rho^{-1}x_2)\zeta(\rho y_1,\rho^{-1}y_2)dxdy \\
&= \frac{1}{\pi}\int\limits_\Pi\int\limits_\Pi \frac{\rho^4 x_2y_2}{(x_1-y_1)^2+\rho^4(x_2+y_2)^2}\zeta(x)\zeta(y)dxdy \\
&\geq \frac{1}{\pi}\int\limits_\Pi\int\limits_{\Pi,|x_1-y_1|<\rho^2} \frac{\rho^4 x_2y_2}{(x_1-y_1)^2+\rho^4(x_2+y_2)^2}\zeta(x)\zeta(y)dxdy \\
&\geq \frac{1}{\pi}\int\limits_\Pi\int\limits_{\Pi,|x_1-y_1|<\rho^2} \frac{x_2y_2}{1+(x_2+y_2)^2}\zeta(x)\zeta(y)dxdy. \\
&\geq \frac{t^2}{\pi}\int\limits_\Pi\int\limits_{\Pi,|x_1-y_1|<\rho^2} \frac{x_2y_2}{1+(x_2+y_2)^2}1_{[0,N]^2}(x)1_{[0,N]^2}(y)dxdy \\
&\geq \frac{t^2}{\pi}\int_0^N\int_0^N \frac{x_2y_2dx_2dy_2}{1+(x_2+y_2)^2}\int\int_{0\leq x_1\leq N, 0\leq y_1\leq N, |x_1-y_1|<\rho^2} dx_1dy_1.
\end{aligned} \tag{2.17}$$

We now set

$$C_N = \frac{t^2}{\pi}\int_0^N\int_0^N \frac{x_2y_2}{1+(x_2+y_2)^2}dx_2dy_2,$$

and

$$J_N(\rho) = \int\int_{0\leq x_1\leq N, 0\leq y_1\leq N, |x_1-y_1|<\rho^2} dx_1dy_1.$$

To calculate this integral, we should describe the area $A(\rho)$ defined by

$$A(\rho) = \{(x_1, y_1) \in [0, N]^2 : |x_1 - y_1| < \rho^2\}.$$

Indeed, if $0 \leq x_1 \leq \rho^2$, then $0 \leq y_1 \leq \rho^2$ or $\rho^2 \leq y_1 < x_1 + \rho^2$, also if $\rho^2 \leq x_1 \leq N - \rho^2$, then $-\rho^2 + x_1 < y_1 < \rho^2 + x_1$ and finally if $N - \rho^2 \leq x_1 \leq N$, then $-\rho^2 + x_1 < y_1 \leq N - \rho^2$ or $N - \rho^2 \leq y_1 \leq N$, therefore we have

$$\begin{aligned} J_N(\rho) &= \int_0^{\rho^2} \int_0^{x_1+\rho^2} dy_1 dx_1 + \int_{\rho^2}^{N-\rho^2} \int_{x_1-\rho^2}^{x_1+\rho^2} dy_1 dx_1 + \int_{N-\rho^2}^{N} \int_{x_1-\rho^2}^{N} dy_1 dx_1 \\ &= \int_0^{\rho^2} (x_1 + \rho^2) dx_1 + 2\rho^2 \int_{\rho^2}^{N-\rho^2} dx_1 + \int_{N-\rho^2}^{k} (-x_1 + N + \rho^2) dx_1 \\ &= \frac{3}{2}\rho^4 + 2\rho^2(N - 2\rho^2) + \frac{3}{2}\rho^4 \\ &= 2N\rho^2 - \rho^4. \end{aligned} \tag{2.18}$$

Hence from (2.17) and (2.18) we get

$$E(v_\rho) \geq C_N(2N\rho^2 - \rho^4). \tag{2.19}$$

Therefore it follows from (2.16) and (2.19) that

$$\Phi_\lambda^n(\zeta_\rho) \geq 2kC_N\rho^2 - C_N\rho^4 - \lambda I_n(\zeta)\rho^n > 0$$

for all small $\rho \in (0, 1]$. □

Lemma 2.15. *Let $n \geq 1$, let $\lambda > 0$, let $k \geq 1$, let $\frac{2k}{2k-1} < p < \infty$ and let $\zeta_0 \in L^1(\Pi) \cap L^p(\Pi)$ be a non-negative function. Let ξ and α be two positive numbers. Then there exists a positive number β depending on α and ξ such that, for every $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ that satisfies $\Phi_\lambda^n(\zeta) \geq \alpha$, $\|\zeta\|_1 = \|\zeta_0\|_1$ and $\|\zeta\|_p = \|\zeta_0\|_p$, there is a square $S(\xi_0)$ of side ξ_0 for which we have*

$$\|\zeta 1_{S(\xi_0)}\|_p > \beta.$$

Proof. Let $1 < q < 2k$ be the conjugate exponent of p. Let $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ be such that $\Phi_\lambda^n(v) \geq \alpha$, $\|\zeta\|_1 = \|\zeta_0\|_1$ and $\|\zeta\|_p = \|\zeta_0\|_p$. Let $\beta > 0$ and assume that for every square $S(\xi_0)$ of side ξ_0

$$\|\zeta 1_{S(\xi_0)}\|_p \leq \beta. \tag{2.20}$$

For fixed $x \in \Pi$, we consider the square S of centre x and side $N\xi_0$, where N is a positive integer greater than $\sqrt{2}\xi_0$. Then we can cover $S \cap \Pi$ by a number N^2 of disjoint squares $\{C_j(\xi_0)\}_{j=1}^{j=N^2} \subset \Pi$ of side ξ_0, so by using Lemma 2.2 we have

$$K\zeta(x) = \frac{1}{2\pi}\left(\int_{S\cap\Pi} + \int_{\Pi\setminus S\cap\Pi}\right) G(x, y)\zeta(y)dy$$

$$\leq \frac{2^{1/k}k}{2\pi}\sum_{j=1}^{j=N^2}\int\limits_{C_j(\xi_0)}\frac{(x_2y_2)^{1/2k}}{|x-y|^{1/k}}\zeta(y)dy+\frac{2^{1/k}k}{2\pi}\int\limits_{|x-y|>\frac{1}{2}N\xi_0}\frac{(x_2y_2)^{1/2k}}{|x-y|^{1/k}}\zeta(y)dy \quad (2.21)$$

We set

$$J_1(x)=\int\limits_{C_j(\xi_0)}\frac{(x_2y_2)^{1/2k}}{|x-y|^{1/k}}\zeta(y)dy \quad\text{and}\quad J_2(x)=\int\limits_{|x-y|>\frac{1}{2}N\xi_0}\frac{(x_2y_2)^{1/2k}}{|x-y|^{1/k}}\zeta(y)dy.$$

By using formula (2.7), we get

$$\begin{aligned} J_2(x) &\leq x_2^{1/2k}\int\limits_{|x-y|>\frac{1}{2}N\xi_0}\frac{\zeta(y)}{|x-y|^{1/2k}}dy+x_2^{1/k}\int\limits_{|x-y|>\frac{1}{2}N\xi_0}\frac{\zeta(y)}{|x-y|^{1/k}}dy \\ &\leq \left((\frac{2x_2}{N\xi_0})^{1/2k}+(\frac{2x_2}{N\xi_0})^{1/k}\right)\|\zeta_0\|_1. \end{aligned} \quad (2.22)$$

Also

$$J_1(x)\leq x_2^{1/2k}\int\limits_{C_j(\xi_0)}\frac{\zeta(y)}{|x-y|^{1/2k}}dy+x_2^{1/k}\int\limits_{C_j(\xi_0)}\frac{\zeta(y)}{|x-y|^{1/k}}dy.$$

Now by Hölder's inequality

$$J_1(x)\leq x_2^{1/2k}\|\zeta 1_{C_j(\xi_0)}\|_p(\int\limits_{C_j(\xi_0)}|x-y|^{-q/2k}dy)^{1/q}+x_2^{1/k}\|\zeta 1_{C_j(\xi_0)}\|_p(\int\limits_{C_j(\xi_0)}|x-y|^{-q/k}dy)^{1/q}.$$

Let D be a disc of centre $x=(0,0)$ and radius $\frac{1}{\sqrt{\pi}}\xi_0$, so $1^*_{C_j(\xi_0)}=1_D$. We then get, by using the basic rearrangement inequality,

$$\int\limits_{C_j(\xi_0)}|x-y|^{-q/2k}dy\leq\int\limits_D|y|^{-q/2k}dy=2\pi\int_0^{\frac{1}{\sqrt{\pi}}\xi_0}r^{\frac{2k-q}{2k}}dr=\frac{4k\pi}{4k-q}(\frac{1}{\sqrt{\pi}}\xi_0)^{\frac{4k-q}{2k}}.$$

The same procedure yields

$$\int\limits_{C_j(\xi_0)}|x-y|^{-q/k}dy\leq\frac{2\pi k}{2k-q}(\frac{1}{\sqrt{\pi}}\xi_0)^{\frac{2k-q}{k}}.$$

Assume now $\frac{1}{\sqrt{\pi}}\xi_0>1$, then by using (2.20) we get

$$\begin{aligned} J_1(x) &\leq \left(x_2^{1/2k}(\frac{4k\pi}{4k-q})^{1/q}(\frac{1}{\sqrt{\pi}}\xi_0)^{\frac{4k-q}{2kq}}+x_2^{1/k}(\frac{2k\pi}{2k-q})^{1/q}(\frac{1}{\sqrt{\pi}}\xi_0)^{\frac{2k-q}{kq}}\right)\|\zeta 1_{S(\xi_0)}\|_p \\ &\leq (x_2^{1/2k}+x_2^{1/k})\beta(\frac{4k\pi}{2k-q})^{1/q}(\frac{1}{\sqrt{\pi}}\xi_0)^{\frac{4k-q}{2kq}}. \end{aligned}$$

Also (2.22) becomes

$$J_2(x) \leq (x_2^{1/2k} + x_2^{1/k}) \frac{2\|\zeta_0\|_1}{(N\xi_0)^{1/2k}}.$$

Therefore it follows from (2.21) that

$$\begin{aligned} K\zeta(x) &\leq \frac{2^{1/k}k}{2\pi} \sum_{j=1}^{j=N^2} (x_2^{1/2k} + x_2^{1/k}) \beta(\frac{1}{\sqrt{\pi}}\xi_0)^{\frac{4k-q}{2kq}} (\frac{4k\pi}{2k-q})^{1/q} \\ &\quad + \frac{2^{1/k}k}{2\pi}(x_2^{1/2k} + x_2^{1/k}) \frac{2\|\zeta_0\|_1}{(N\xi_0)^{1/2k}} \\ &\leq \frac{2^{1/k}k}{2\pi} \left(N^2 \beta(\frac{1}{\sqrt{\pi}}\xi_0)^{\frac{4k-q}{2kq}} (\frac{4k\pi}{2k-q})^{1/q} + \frac{2\|\zeta_0\|_1}{(N\xi_0)^{1/2k}} \right) (x_2^{1/2k} + x_2^{1/k}). \end{aligned}$$

By setting

$$M = (\frac{1}{\sqrt{\pi}}\xi_0)^{\frac{4k-q}{2kq}} (\frac{4k\pi}{k-q})^{1/q} \quad \text{and} \quad A(N,\beta) = M\beta N^2 + \frac{2\|\zeta_0\|_1}{(N\xi_0)^{1/2k}},$$

then follows from the definition of Φ_λ^n that

$$\begin{aligned} \Phi_\lambda^n(\zeta) &\leq \frac{2^{1/k}k}{4\pi} A(N,\beta) \int_\Pi \left(x_2^{1/2k} + x_2^{1/k} - \frac{4\lambda\pi}{2^{1/k}knA(N,\beta)} x_2^n \right) \zeta(x)dx \\ &\leq \frac{2^{1/k}k}{4\pi} \|\zeta\|_1 A(N,\beta) (\max_{x_2>0}(x_2^{1/2k} - x_2^{1/k}) + \max_{x_2>0}(2x_2^{1/k} - \frac{4\lambda\pi}{2^{1/k}knA(N,\beta)} x_2^n)) \\ &= \frac{2^{1/k}k}{4\pi} \|\zeta\|_1 (M_1 A(N,\beta) + M_2 (A(N,\beta))^{\frac{kn-1}{kn}}), \end{aligned} \tag{2.23}$$

where M_1 and M_2 are two positive constants depending on k and λ. We set

$$m = \min\{\frac{2\alpha\pi}{2^{1/k}kM_1\|\zeta_0\|_1}, (\frac{2\alpha\pi}{2^{1/k}kM_2\|\zeta_0\|_1})^{\frac{kn}{kn-1}}\}.$$

Now we can choose N large enough so that $2\|\zeta_0\|_1 (N\xi_0)^{-1/2k} \leq m/2$, and also we choose β small enough to ensure that $M\beta N^2 \leq m/2$. Hence we find that $A(N,\beta) \leq m$, and therefore from (2.23) we get $\Phi_\lambda^n(\zeta) \leq \alpha$. But this is a contradiction; thus for some $\beta > 0$ there is a square $S(\xi_0)$ of side ξ_0 for which we have

$$\|\zeta 1_{S(\xi_0)}\|_p > \beta.$$

This completes the proof. □

Lemma 2.16. *Let the assumptions about n, k, p, λ, ξ_0, α be the same as in Lemma 2.15. Let $\zeta_0 \in L^p(\Pi)$ be a non-negative function having support of finite measure. Let β be the number provided by Lemma 2.15. Then there exists a positive number γ depending on β such that, for every $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ with $\Phi_\lambda^n(\zeta) \geq \alpha$, $\|\zeta\|_1 = \|\zeta_0\|_1$ and $\|\zeta\|_p = \|\zeta_0\|_p$,* *there is a square $S(\xi_0)$ of side ξ_0 for which*

$$|\{y \in S(\xi_0) | \zeta(y) > 0\}| \geq \gamma.$$

Furthermore, there exists $\eta > 0$ depending on γ such that

$$\int_{S(\xi_0)} y_2 \zeta(y) dy \geq \eta.$$

Proof. Let $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ satisfy $\Phi^n_\lambda(\zeta) \geq \alpha$, $\|\zeta\|_1 = \|\zeta_0\|_1$ and $\|\zeta\|_p = \|\zeta_0\|_p$. Then by choice of $\beta > 0$ in Lemma 2.15, there exists square $S(\xi_0) \subset \Pi$ of side ξ_0, such that

$$\int_{S(\xi_0)} |\zeta(x)|^p dx > \beta^p.$$

By setting $V = \{y \in S(\xi_0) | \zeta(y) > 0\}$, then we have

$$\beta^p < \int_{S(\xi_0)} |\zeta(y)|^p dy = \int_{\Pi} |\zeta(y)|^p 1_V(y) dy.$$

Let $\zeta^\triangle$ and $(1_V)^\triangle$ be the decreasing rearrangements of ζ and 1_V respectively. Then

$$\int_{\Pi} |\zeta(y)|^p 1_V(y) dy \leq \int_0^\infty |\zeta^\triangle(t)|^p (1_V)^\triangle(t) dt = \int_0^{|V|} |\zeta^\triangle(t)|^p dt = \int_0^{|V|} |\zeta_0^\triangle(t)|^p dt,$$

where $\zeta_0^\triangle$ is the decreasing rearrangement of ζ_0. Therefore we find that

$$\int_0^{|V|} |\zeta_0^\triangle(t)|^p dt > \beta^p.$$

Now by Lebesgue's dominated convergence theorem [27, Theorem D], we have

$$\int_0^\tau |\zeta_0^\triangle(t)|^p dt \to 0 \quad \text{as} \quad \tau \to 0,$$

then there exists $\gamma > 0$ such that $\int_0^\gamma |\zeta_0^\triangle(t)|^p dt < \beta^p$. Therefore $|V| \geq \gamma$ that is

$$|\{y \in S(\xi_0) | \zeta(y) > 0\}| \geq \gamma.$$

Now let δ be a positive number chosen such that $\delta \xi_0 < \frac{1}{2}\gamma$, and let R be the rectangle defined by

$$R = [z_1 - \xi_0/2, z_1 + \xi_0/2] \times [z_2 - \xi_0/2 + \delta, z_2 + \xi_0/2],$$

where $z = (z_1, z_2)$ is the centre of $S(\xi_0)$. Then for all $y \in \Pi$ we have

$$y_2 1_{S(\xi_0)}(y) \geq y_2 1_D(y) \geq \delta 1_D(y),$$

where $D = R \cap \operatorname{supp} \zeta_0$. Thus

$$\int\limits_{S(\xi_0)} y_2 \zeta(y) dy \geq \int\limits_{S(\xi_0)} \delta 1_D(y) \zeta(y) dy = \delta \int\limits_{D \cap S(\xi_0)} \zeta(y) dy.$$

Since $D \cap S(\xi_0) = D$, then we get

$$\int\limits_{D \cap S(\xi_0)} \zeta(y) dy = \int_D \zeta(y) dy \geq \int_0^{|D|} \zeta^{\triangledown}(t) dt = \int_0^{|D|} \zeta_0^{\triangledown}(t) dt,$$

where $\zeta^{\triangledown}$ and $\zeta_0^{\triangledown}$ are the increasing rearrangements of ζ and ζ_0 respectively on $[0, |\operatorname{supp} \zeta_0|]$. Since $|D| \geq \frac{1}{2}\gamma$, then it follows that

$$\int\limits_{S(\xi_0)} y_2 \zeta(y) dy \geq \eta,$$

where $\eta = \delta \int_0^{\frac{1}{2}\gamma} \zeta_0^{\triangledown}(t) dt$. □

Lemma 2.17. *Let $\alpha > 0$, let $X > 1$, let $k \geq 1$, let $\frac{2k}{2k-1} < p < \infty$ and let $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ be a non-negative function. Let $\{X_i\}_{i=0}^{i=i_*}$ be a positive sequence in $\mathbb{R}$ such that $X_i \leq X_j$ for $i \leq j$ and $X_{i_*} = X$. Then there exists a positive number m depending on $\|\zeta\|_1$, $\|\zeta\|_p$, k and X such that if ζ satisfies*

$$\int\limits_{x_2 < X, |x_1| < X_0} \zeta(x) K\zeta(x) dx \geq \alpha,$$

then we have

$$\int\limits_{|x_1| < X_0, x_2 < X_0} x_2 \zeta(x) dx + \sum_{i=1}^{i=i_*} \int\limits_{X_{i-1} \leq x_2 < X_i, |x_1| < X_0} \zeta(x) dx \geq \frac{\alpha}{m}.$$

Proof. We write

$$\int\limits_{|x_1| < X_0, x_2 < X} \zeta(x) K\zeta(x) dx = \left(\int\limits_{|x_1| < X_0, x_2 < X_0} + \sum_{i=1}^{i=i_*} \int\limits_{|x_1| < X_0, X_{i-1} \leq x_2 \leq X_i} \right) \zeta(x) K\zeta(x) dx. \tag{2.24}$$

Applying the Mean Value inequality as in [13, Lemma 3] with the estimate of Lemma 2.6, we can show that, there exists a positive number N depending on p such that

$$K\zeta(x) \leq N(\|\zeta\|_p + \|\zeta\|_1) x_2.$$

Hence we have

$$\int\limits_{|x_1|<X_0, x_2<X_0} \zeta(x)K\zeta(x)dx \leq N(\|\zeta\|_1 + \|\zeta\|_p) \int\limits_{|x_1|<X_0, x_2\leq X_0} x_2\zeta(x)dx. \tag{2.25}$$

Also by Lemma 2.4, there exists a positive number C depending on p and k such that

$$K\zeta(x) \leq C(\|\zeta\|_1 + \|\zeta\|_p)(x_2^{1/2k} + x_2^{1/k}).$$

Hence it follows that

$$\int\limits_{|x_1|<X_0, X_{i-1}\leq x_2\leq X_i} \zeta(x)K\zeta(x)dx \leq C(\|\zeta\|_1 + \|\zeta\|_p)X^{1/k} \int\limits_{|x_1|<X_0, X_{i-1}\leq x_2\leq X_i} \zeta(x)dx. \tag{2.26}$$

It follows then from (2.24), (2.25) and (2.26) that

$$\int\limits_{|x_1|<X_0, x_2<X} \zeta(x)K\zeta(x)dx \leq m\left(\int\limits_{|x_1|<X_0, x_2<X_0} x_2\zeta + \sum_{i=1}^{i=i_*} \int\limits_{|x_1|<X_0, X_{i-1}\leq x_2\leq X_i} \zeta\right), \tag{2.27}$$

where

$$m = \max\{N, CX^{1/k}\}(\|\zeta\|_1 + \|\zeta\|_p).$$

Therefore from (2.27), we get

$$\int\limits_{|x_1|<X_0, x_2<X_0} x_2\zeta(x)dx + \sum_{i=1}^{i=i_*} \int\limits_{|x_1|<X_0, X_{i-1}\leq x_2<X_i} \zeta(x)dx \geq \frac{\alpha}{m}.$$

Thus the result follows. □

2.5 Proofs of our main results

In this section, all the results which were proved in section 2.3 and 2.4 will be used to prove the main theorem. Before that, we state a result concerning a lower bound for the measure of the set

$$\{x \in \Pi | K\zeta(x) - \frac{\lambda}{n}x_2^n > 0\}$$

whenever $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ $(p > 1)$ is independent of λ.

Lemma 2.18. *Let $n \geq 1$, let $\lambda > 0$, let $p > 1$ and let $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ be a non-negative function independent of λ. Then for all $\lambda \in (0, \frac{nC(\zeta)}{2\pi})$, we have*

$$|\{x \in \Pi | K\zeta(x) - \frac{\lambda}{n}x_2^n > 0\}| \geq \frac{\pi}{2}((\frac{nC(\zeta)}{2\pi\lambda})^{1/n} - 1),$$

where

$$C(\zeta) = \int_{\Pi} \frac{y_2}{1+|y|^2}\zeta(y)dy.$$

Proof. By using Lemma 2.3, for all $x \in \Pi$ we have

$$K\zeta(x) \geq \frac{x_2 C(\zeta)}{2\pi(1+|x|^2)}.$$

Since $x_2 < 1 + |x|^2$, then for all $n \geq 1$

$$K\zeta(x) \geq \frac{x_2^n C(\zeta)}{2\pi(1+|x|^2)^n};$$

hence we find that

$$K\zeta(x) - \frac{\lambda}{n}x_2^n \geq (\frac{C(\zeta)}{2\pi(1+|x|^2)^n} - \frac{\lambda}{n})x_2^n.$$

Therefore we have

$$\begin{aligned} |\{x \in \Pi | K\zeta(x) - \frac{\lambda}{n}x_2^n > 0\}| &\geq |\{x \in \Pi | |x|^2 < (\frac{nC(\zeta)}{2\pi\lambda})^{1/n} - 1\}| \\ &= \frac{\pi}{2}((\frac{nC(\zeta)}{2\pi\lambda})^{1/n} - 1). \end{aligned}$$

This completes the proof. □

Proof of Theorem 2.1

Let $\lambda > 0$, $n \geq 2$ fixed, let $p > 2$ and let q be the conjugate exponent of p. Let $a > 0$ be such that $|\{x \in \Pi|\zeta_0(x) \geq 0\}| = \pi a^2$. Let X be a positive number chosen as in Lemma 2.11. Let $\mathcal{F}(\zeta_0)$ be the set of rearrangements of ζ_0 on Π. Let $\mathcal{R}(\zeta_0)$ be the set defined as in Lemma 2.13 and let $\mathcal{R}^s(\zeta_0)$ be the set of Steiner-symmetric function in $\mathcal{R}(\zeta_0)$. By Lemma 2.13, a maximising sequence $\{\zeta_j\}_{j=1}^{\infty}$ for Φ_λ^n relative to $\mathcal{F}(\zeta_0)$, can be chosen in $\mathcal{R}(\zeta_0)$. Henceforth, we assume that $\zeta_j \in \mathcal{R}(\zeta_0)$. By Lemma 2.11, for each j, we can choose f_j having bounded support such that

1. $f_j \in \mathcal{R}(\zeta_0)$
2. $\text{supp} f_j \subset \mathbb{R} \times (0, X)$
3. $\Phi_\lambda^n(f_j) \geq \Phi_\lambda^n(\zeta_j)$.

We now let $\{\xi_j\}_{j=1}^{\infty}$ denote a sequence that satisfies $\xi_j \geq j$ and supp $f_j \subset \Pi(\xi_j, X)$ for all $j \in \mathbb{N}$, also we define the sets

$$\mathcal{J}(\xi_j, X) = \{\zeta \in \mathcal{R}(\zeta_0) | \text{supp}\, \zeta \subset \Pi(\xi_j, X)\}.$$

The symmetry of G, and Lemmas 2.7 and 2.8 show that

$$K : L^p(\Pi(\xi_j, X)) \to L^q(\Pi(\xi_j, X))$$

is a compact, symmetric and strictly positive operator. Therefore the functional Φ_λ^n is weakly sequentially continuous and strictly convex on $L^p(\Pi(\xi_j, X))$, hence by using Theorem 1.1, the functional Φ_λ^n has a maximiser relative to $\mathcal{J}(\xi_j, X)$ for all $\lambda > 0$ and $n \geq 2$. By using Lemma 2.9, we have

$$\Phi_\lambda^n(\zeta^s) \geq \Phi_\lambda^n(\zeta).$$

Therefore there exists a maximiser for Φ_λ^n relative to $\mathcal{J}(\xi_j, X)$ that belongs to the sets

$$\mathcal{J}^s(\xi_j, X) = \{\zeta \in \mathcal{R}^s(\zeta_0) | \text{supp}\, \zeta \subset \Pi(\xi_j, X)\}.$$

Now let $\{\bar{\zeta}_j\}_{j=1}^\infty$ be a sequence of maximisers for the functional Φ_λ^n relative to the sets $\mathcal{J}^s(\xi_j, X)$; then it follows that $\Phi_\lambda^n(\bar{\zeta}_j) \geq \Phi_\lambda^n(f_j)$. Hence $\{\bar{\zeta}_j\}_{j=1}^\infty$ is a maximising sequence for Φ_λ^n relative to $\mathcal{R}(\zeta_0)$, and then by using Theorem 1.1 again, there is an increasing function ϕ_j such that

$$\bar{\zeta}_j = \phi_j \circ (K\bar{\zeta}_j - \frac{\lambda}{n} x_2^n) \tag{2.28}$$

almost everywhere in $\Pi(\xi_j, X)$, for fixed $\lambda > 0$ and n. We now proceed to show that if $n \geq 3$ and $\lambda > 0$, then for all j large enough, the support of $\bar{\zeta}_j$ is contained in the region where $K\bar{\zeta}_j - \frac{\lambda}{n} x_2^n > 0$. Indeed, by Lemma 2.14 we can choose $\alpha > 0$ as

$$\alpha = \Phi_\lambda^n(f) > 0$$

for some function $f \in \mathcal{R}(\zeta)$ having bounded support, where $\lambda > 0$ and $n \geq 3$. Therefore we can choose $j_0 \in \mathbb{N}$ which may depend on λ, such that

$$\Phi_\lambda^n(\bar{\zeta}_j) \geq \alpha$$

for all $j \geq j_0$. It follows from Lemma 2.15 that there are two positive numbers β and ξ_0, satisfying

$$\int_{S(\xi_0)} |\bar{\zeta}_j(x)|^p dx > \beta^p,$$

where $S(\xi_0)$ is some square of side ξ_0. Since $\bar{\zeta}_j$ is Steiner-symmetric, then we can assume that $S(\xi_0) \subset \mathbb{R} \times (0, X)$ is symmetric in the x_2 axis. Hence by using Lemma 2.16, there exists a positive number η independent of j such that

$$\int_{S(\xi_0)} x_2 \bar{\zeta}_j(x) dx \geq \eta \tag{2.29}$$

for all $j \geq j_0$. We now set

$$C(\xi_0) = \min_{S(\xi_0)} \frac{1}{1 + |x|^2},$$

then by using Lemma 2.2 and (2.29), we find

$$K\bar{v}_j(x) \geq \frac{C(\xi_0)\eta x_2}{2\pi(1+|x|^2)}. \tag{2.30}$$

For all $x \in \Pi$ we have

$$1+|x|^2 \quad \leq \quad (1+x_2^2)(1+x_1^2). \tag{2.31}$$

Thus by applying (2.31), the formula in (2.30) becomes

$$K\bar{\zeta}_j(x) \geq \frac{Nx_2}{(1+x_2^2)(1+x_1^2)},$$

where $N = \frac{\eta C(\xi_0)}{4\pi}$. Therefore it follows that

$$K\bar{\zeta}_j(x) - \frac{\lambda}{n}x_2^n \geq \left(\frac{N}{(1+x_2^2)(1+x_1^2)} - \frac{\lambda}{n}x_2^{n-1}\right)x_2.$$

Now the region defined by the inequality

$$|x_1| < \left(\frac{Nn}{\lambda(1+x_2^2)x_2^{n-1}} - 1\right)^{1/2} \quad \text{and} \quad x_2 < X$$

has infinite measure for all $n \geq 3$ with respect to μ measure. Hence we can choose $\varepsilon > 0$ independent of j such that

$$|\{x \in \mathbb{R} \times (0, X) | x_1|^2 < \frac{Nn}{(1+x_2^2)(n\varepsilon + \lambda x_2^{n-1})} - 1\}| \geq \pi a^2.$$

We now set

$$V(\xi_j) = \{x \in \Pi | \bar{\zeta}_j(x) > 0\}$$

and

$$R(\xi_j, X) = \{x \in \Pi(\xi_j, X) | K\bar{\zeta}_j(x) - \frac{\lambda}{n}x_2^n \geq \varepsilon\}.$$

The fact that $\bar{\zeta}_j$ is an increasing function of $K\bar{\zeta}_j(x) - \frac{\lambda}{n}x_2^n$ on $\Pi(\xi_j, X)$ and (2.28) imply that apart from a set of zero measure

$$V(\xi_j) \subset R(\xi_j, X).$$

Now Lemma 2.5 shows that all points of $R(\xi_j, X)$ satisfy

$$M\|\zeta_0\|_p(x_2^{1/2k} + x_2^{1/k})\min\{1, |x_1|^{\frac{-1}{2k+p}}\} - \frac{\lambda}{n}x_2^n \geq \varepsilon,$$

where M is a positive constant independent of λ and j. Therefore

$$M\|\zeta_0\|_p(x_2^{1/2k}+x_2^{1/k})|x_1|^{\frac{-1}{2k+p}}-\frac{\lambda}{n}x_2^n\geq\varepsilon$$

for all $x\in R(\xi_j,X)$. Hence there exists $X_1>0$ independent of j such that $1<|x_1|<X_1$ for all $x\in R(\xi_j,X)$. We set $r=\max\{\xi_{j_0},X_1,X\}$ and let $j_*\in\mathbb{N}$ be such that $\xi_j\geq r$ for all $j\geq j_*$, then the support of $\bar{\zeta}_j$ is bounded by the rectangle $(-X_1,X_1)\times(0,X)\subset\Pi(r,X)$, so $\bar{\zeta}_j\in\mathcal{J}(\xi_{j_*},X)$ for all $j\geq j_*$. It follows that $\bar{\zeta}_{j_*}$ maximises the functional Φ_λ^n relative to $\mathcal{J}(\xi_j,X)$ for all $j\geq j_*$, so $\bar{\zeta}_{j_*}$ maximises Φ_λ^n relative to $\mathcal{R}(\zeta_0)$. Therefore $\bar{\zeta}_{j_*}$ maximises $\Phi_\lambda^n(\zeta)$ relative to $\mathcal{F}(\zeta_0)$. By writing $\bar{\zeta}_{j_*}=\zeta$ then from (2.28) we get

$$\zeta=\phi_{j_*}\circ(K\zeta-\frac{\lambda}{n}x_2^n) \tag{2.32}$$

almost everywhere in $\Pi(r,X)$. It remains only to extend (2.32) to $\Pi\backslash\Pi(r,X)$. We can assume that $\phi_{j_*}(t)>0$ for $t\in\mathrm{dom}(\phi_{j_*})$ and we consider the function ϕ defined by

$$\phi(t)=\begin{cases}\phi_{j_*}(t) & \text{if } t\geq\varepsilon\\ 0 & \text{if } t<\varepsilon,\end{cases}$$

where $dom(\phi_{j_*})$ is the domain of definition of ϕ_{j_*}. Since $K\zeta(x)-\frac{\lambda}{n}x_2^n<\varepsilon$ outside $\Pi(r,X)$ and ϕ_{j_*} is an increasing function of $K\zeta(x)-\frac{\lambda}{n}x_2^n$ almost everywhere in $\Pi(r,X)$, then ϕ is an increasing function in Π, hence

$$\zeta=\phi\circ(K\zeta-\frac{\lambda}{n}x_2^n).$$

almost everywhere in Π. Therefore by setting $\psi=:K\zeta$ we have

$$-\Delta\psi=\phi\circ(\psi-\frac{\lambda}{n}x_2^n)$$

almost everywhere in Π for some increasing function ϕ. This completes the discussion of the case of general $\lambda>0$ and $n\geq3$. □

We now consider the case when $n=2$. We proved the functional Φ_λ^n attains a maximum value relative to $\mathcal{J}(\xi_j,X)$ for all $\lambda>0$ and $n\geq2$, so for $n=2$ we need just to show that if $\bar{\zeta}_j$ is a sequence of maximiser of Φ_λ^2 relative to $\mathcal{J}(\xi_j,X)$, then the support of $\bar{\zeta}_j$ is contained in the set where $K\bar{\zeta}_j-\frac{\lambda}{2}x_2^2>0$, provided λ is small and fixed. Indeed, by using [13, Lemma 3], there exists a positive constant N such that

$$K\bar{\zeta}_j(x)\leq N\|\zeta_0\|_p x_2. \tag{2.33}$$

Now from Lemma 2.10, for fixed j we have

$$\sup_{\zeta\in\mathcal{J}^s(\xi_j,X)}E(\zeta)\to\infty\quad\text{as}\quad\lambda\to0.$$

Hence we can choose $j_0 \in \mathbb{N}$ and $\Lambda > 0$ so small such that

$$\sup_{\zeta \in \mathcal{J}^s(\xi_j, X)} \Phi^2_\Lambda(\zeta) \geq 7aN\|\zeta_0\|_p. \tag{2.34}$$

Then it follows from (2.34) that

$$\sup_{x \in \Pi(\xi_j, X)} (K\bar{\zeta}_j(x) - \frac{\lambda}{2}x_2^2) > 7aN$$

for all $j \geq j_0$ and $\lambda \in (0, \Lambda)$.

Henceforth, we assume that $j \geq j_0$ and $\lambda \in (0, \Lambda)$. Let $z \in \Pi(\xi_j, X)$ be such that

$$K\bar{\zeta}_j(z) - \frac{\lambda}{2}z_2^2 \geq K\bar{\zeta}_j(x) - \frac{\lambda}{2}x_2^2$$

for all $x \in \Pi(\xi, X)$. Then we find that

$$K\bar{\zeta}_j(z) - \frac{\lambda}{2}z_2^2 > 7aN. \tag{2.35}$$

By using (2.33) we get

$$Nz_2 - \frac{\lambda}{2}z_2^2 \geq 7aN.$$

Since $\lambda > 0$, then we have $z_2 > 7a$, and therefore the rectangle $\Pi(\xi_j, X)$ contains at least a quadrant D of the half disc

$$\{x \in \Pi | x - z| < 4a, x_2 < z_2\},$$

with $|D| = 4\pi a^2$. By using Lemma 2.6 and the Mean Value Inequality, for all $x \in D$, $j \geq j_0$ and $\lambda \in (0, \Lambda)$ we have

$$\begin{aligned} K\bar{\zeta}_j(x) - \frac{\lambda}{2}x_2^2 &\geq K\bar{\zeta}_j(z) - \frac{\lambda}{2}z_2^2 - N|x - z| - \frac{\lambda}{2}(x_2^2 - z_2^2) \\ &\geq K\bar{\zeta}_j(z) - \frac{\lambda}{2}z_2^2 - 4aN \\ &\geq 3aN. \end{aligned} \tag{2.36}$$

Thus we obtained that the set

$$R(\xi_j, X) = \{x \in \Pi(\xi_j, X) | K\bar{\zeta}_j(x) - \frac{\lambda}{2}x_2^2 \geq aN\}$$

has measure greater than πa^2 for all $j \geq j_0$ and $\lambda \in (0, \Lambda)$. Since $\bar{\zeta}_j$ is essentially an increasing function of $K\bar{\zeta}_j(x) - \frac{\lambda}{2}x_2^2$, we have

$$V(\xi_j) \subset R(\xi_j, X)$$

apart from a set of zero measure.

Now Lemma 2.5 shows that all points of $R(\xi_j, X)$ satisfy

$$M\|\zeta_0\|_p(x_2^{1/k}+x_2^{1/2k})\min\{1,|x_1|^{\frac{-1}{2k+p}}\}-\frac{\lambda}{2}x_2^2\geq aN.$$

Hence there exists a positive constant $Y(\lambda)$ independent of j such that $|x_1|<Y(\lambda)$ for all $x\in R(\xi_j,X)$. For all $\lambda\in(0,\Lambda)$, we set $r(\lambda)=\max\{X,Y(\lambda)\}$. Let j_* be such that $\xi_j\geq r(\lambda)$ for all $j\geq j_*$. Then by using the same argument as in the case when $n\geq 3$, we find $\bar{\zeta}_{j_*}$ maximises the functional Φ_λ^2 relative to $\mathcal{F}$ for all λ positive and small. By writing $\bar{\zeta}_j$, then as before we have

$$-\Delta\psi=\phi\circ(\psi-\frac{\lambda}{2}x_2^2)$$

almost everywhere in Π for some increasing function ϕ and λ small and positive. □

2.6 An alternative proof of Theorem 2.1 when $n=2$

From the last section, we deduce that the most important stage in the proof of the Theorem is to prove that if $\bar{\zeta}_j$ is a maximiser for Φ_λ^2 relative to $\mathcal{J}(\xi_j,X)$, then the set where $K\bar{\zeta}_j(x)-\frac{\lambda}{2}x_2^2>0$ has a positive measure which is greater than πa^2. However, in this section, we use a certain estimate for the function $K\bar{\zeta}_j$, to show that

$$|\{x\in\Pi(\xi_j,X)|K\bar{\zeta}_j(x)-\frac{\lambda}{2}x_2^2>0\}|\to\infty\quad\text{as}\quad\lambda\to 0.$$

Hence we prove the measure of the set where $K\bar{\zeta}_j(x)-\frac{\lambda}{2}x_2^2>0$ is greater than πa^2 for all j large and λ small. This estimate can be given as follows.

Lemma 2.19. *Let $0<a<\infty$, let $X>0$, let $k>4$, let $2<p<\infty$ and let $\zeta\in L^p(\Pi_X)$ be Steiner-symmetric having support of area πa^2, where $\Pi_X=\mathbb{R}\times(0,X)$. There exists a positive constant M depending on p and k such that if $|x_1|>2$, then we have*

$$K\zeta(x)\leq M\|\zeta\|_p(\frac{X}{|x_1|})^{2/k}.$$

Proof. For $|x_1|>2$, we set

$$H_1(x)=\int\limits_{|x_1-y_1|<1}G(x,y)\zeta(y)dy,$$

$$H_2(x)=\int\limits_{1\leq|x_1-y_1|\leq\frac{1}{2}|x_1|}G(x,y)\zeta(y)dy$$

and

$$H_3(x)=\int\limits_{|x_1-y_1|>\frac{1}{2}|x_1|}G(x,y)\zeta(y)dy.$$

If $x_2 < X$ and $y_2 < X$, then by Lemma 2.2 we have

$$G(x,y) \le \frac{2^{2/k}k(x_2y_2)^{1/k}}{2\pi|x-y|^{2/k}} \le \frac{kX^{2/k}}{\pi|x-y|^{2/k}},$$

where $k > 4$. Hence it follows that

$$H_3(x) \le \frac{kX^{2/k}}{\pi} \int\limits_{|x_1-y_1|>\frac{1}{2}|x_1|} \frac{\zeta(y)}{|x-y|^{2/k}}dy \le \frac{2^{2/k}k}{\pi}(\frac{X}{|x_1|})^{2/k}\|\zeta\|_1. \tag{2.37}$$

We recall now that if ζ is Steiner symmetric, then

$$V(x_1) := \int_0^\infty \zeta(x_1,x_2)dx_2 \le \frac{\|\zeta\|_1}{|x_1|}.$$

Hence by setting $x_1 - y_1 = \pm t$ we have

$$\begin{aligned} H_2(x) &\le \frac{kX^{2/k}}{\pi} \int\limits_{1\le|x_1-y_1|\le\frac{1}{2}|x_1|} \frac{\zeta(y)}{|x-y|^{2/k}}dy \\ &\le \frac{kX^{2/k}}{\pi}\left(\int_1^{\frac{1}{2}|x_1|}\int_0^\infty \frac{\zeta(t+x_1,y_2)}{t^{2/k}}dy_2dt + \int_1^{\frac{1}{2}|x_1|}\int_0^\infty \frac{\zeta(-t+x_1,y_2)}{t^{2/k}}dy_2dt\right) \\ &= \frac{kX^{2/k}}{\pi}\left(\int_1^{\frac{1}{2}|x_1|} V(t+x_1)\frac{dt}{t^{2/k}} + \int_1^{\frac{1}{2}|x_1|} V(-t+x_1)\frac{dt}{t^{2/k}}\right) \\ &\le \frac{kX^{2/k}\|\zeta\|_1}{\pi}\int_0^{\frac{1}{2}|x_1|}(\frac{1}{|t+x_1|}+\frac{1}{|-t+x_1|})\frac{dt}{t^{2/k}}. \end{aligned} \tag{2.38}$$

Since $|x_1| > 2$ and $|t| < \frac{1}{2}|x_1|$, then it follows that $|t+x_1| \ge |-t+x_1| \ge \frac{1}{2}|x_1|$ if $x_1 \ge 0$ and $|-t+x_1| \ge |t+x_1| \ge \frac{1}{2}|x_1|$ if $x_1 < 0$. Therefore from (2.38), we find

$$H_2(x) \le \frac{2^{2/k}k^2}{(k-2)\pi}\|\zeta\|_1(\frac{X}{|x_1|})^{2/k}. \tag{2.39}$$

Now since $|x-y|^{-2/k}$ is a decreasing function with respect to $|x_2-y_2|$, then by applying the classical rearrangement inequality we have

$$\int\limits_{|x_1-y_1|<1} \frac{\zeta(y)}{|x-y|^{2/k}}dy \le \int\limits_{|x_1-y_1|<1} \frac{\zeta_*(y)}{|x-y|^{2/k}}dy, \tag{2.40}$$

where $\zeta_*(x_1,.)$ is the rearrangement of $\zeta(x_1,.)$ that is symmetric decreasing about x_2. Now if $y \in \operatorname{supp}\zeta_*$, then it follows that $|x_2-y_2| \le \frac{1}{4}(\pi a^2)|y_1|^{-1}$. Hence if $|x_1-y_1| < 1$, then we get $|y_1|^{-1} < 2|x_1|^{-1}$ for all $|x_1| > 2$. Therefore by using (2.40) we have

$$\begin{aligned}
\int\limits_{|x_1-y_1|<1} \frac{\zeta(y)}{|x-y|^{2/k}}dy &\leq \int\limits_{|x_1-y_1|<1, |x_2-y_2|\leq\frac{\pi}{4}a^2|y_1|^{-1}} \frac{\zeta_*(y)}{|x-y|^{2/k}}dy \\
&\leq \int\limits_{|x_1-y_1|<1, |x_2-y_2|\leq\frac{\pi}{2}a^2|x_1|^{-1}} \frac{\zeta_*(y)}{|x-y|^{2/k}}dy. \qquad (2.41)
\end{aligned}$$

By Hölder's inequality

$$\begin{aligned}
\int\limits_{|x_1-y_1|<1, |x_2-y_2|\leq\frac{\pi}{2}a^2|x_1|^{-1}} \frac{\zeta_*(y)}{|x-y|^{2/k}}dy &\leq \|\zeta\|_p\Big(\int\limits_{|x_1-y_1|<1, |x_2-y_2|\leq\frac{\pi}{2}a^2|x_1|^{-1}} \frac{dy}{|x-y|^{2q/k}}\Big)^{1/q} \\
&= \|\zeta\|_p\Big(\int_{-\frac{\pi}{2}a^2|x_1|^{-1}}^{\frac{\pi}{2}a^2|x_1|^{-1}} \int_{-1}^{1} \frac{dtds}{(s^2+t^2)^{q/k}}\Big)^{1/q} \\
&\leq 2\|\zeta\|_p\Big(\frac{\pi}{2}a^2|x_1|^{-1}\int_0^1 \frac{dt}{t^{2q/k}}\Big)^{1/q} \\
&= 2\|\zeta\|_p\Big(\frac{\pi a^2 k}{2(k-2q)}\Big)^{1/q}|x_1|^{-1/q}.
\end{aligned}$$

Since $|x_1| > 2$ and $q < k/2$, then it follows that

$$\int\limits_{|x_1-y_1|<1, |x_2-y_2|\leq\frac{\pi}{2}a^2|x_1|^{-1}} \frac{\zeta_*(y)}{|x-y|^{2/k}}dy \leq 2\|\zeta\|_p\Big(\frac{\pi a^2 k}{2(k-2q)}\Big)^{1/q}|x_1|^{-2/k}. \qquad (2.42)$$

Therefore by using (2.41) and (2.42) we get

$$H_1(x) \leq \frac{2k}{\pi}\Big(\frac{\pi a^2 k}{2(k-2q)}\Big)^{1/q}\|\zeta\|_p\Big(\frac{X}{|x_1|}\Big)^{2/k}. \qquad (2.43)$$

Recall now that

$$\|\zeta\|_1 \leq (\pi a^2)^{1/q}\|\zeta\|_p.$$

Also we write

$$K\zeta(x) = H_1(x) + H_2(x) + H_3(x).$$

Then by using (2.37), (2.39) and (2.43) we find that there exists a positive constant M depending on a, p and k such that if $|x_1| > 2$, then have

$$K\zeta(x) \leq M\|\zeta\|_p\Big(\frac{X}{|x_1|}\Big)^{2/k}.$$

This completes the proof. $\square$

2.6.1 Second proof of Theorem 2.1 when $n = 2$

Let $k > \frac{2(4-\tau)}{2-\tau}$, where $\tau = (\frac{1}{2})^8$. Let $\bar{\zeta}_j$ be a maximiser for the functional Φ_λ^2 relative to $\mathcal{J}(\xi_j, X)$. Let λ_0 be a positive number chosen so small that

$$\alpha = \Phi_{\lambda_0}^2(\zeta_1) > 0,$$

where ζ_1 is a rearrangement, having bounded support, of ζ_0 in $\mathbb{R} \times (0, X)$. Then there exists a positive $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ and $0 < \lambda < \lambda_0$ we have

$$\Phi_\lambda^2(\bar{\zeta}_j) \geq \alpha;$$

hence it follows that

$$\int_{x_2 < X} \bar{\zeta}_j(x) K\bar{\zeta}_j(x) dx \geq 2\alpha$$

for all $j \geq j_0$ and $0 < \lambda < \lambda_0$. Now by Lemma 2.19 we have

$$K\bar{\zeta}_j(x) \leq M\|\zeta_0\|_p (\frac{X}{|x_1|})^{2/k}$$

for all $|x_1| > 2$, where M is positive constant independent of λ and j. Thus for $\epsilon > 0$ independent of λ and j we get

$$\int_{x_2 < X, |x_1| > \epsilon X} \bar{\zeta}_j(x) K\bar{\zeta}_j(x) dx \leq M\|\zeta_0\|_1 \|\zeta_0\|_p \epsilon^{-2/k}.$$

Therefore we can take $\epsilon_0 = (\frac{M\|\zeta_0\|_1\|\zeta_0\|_p}{\alpha})^{k/2}$ to ensure that if $\epsilon \geq \epsilon_0$, then

$$\int_{x_2 < X, |x_1| > \epsilon X} \bar{\zeta}_j(x) K\bar{\zeta}_j(x) dx \leq \alpha$$

Thus for all $j \geq j_0$, $0 < \lambda < \lambda_0$ and $\epsilon \geq \epsilon_0$ we have

$$\int_{x_2 < X, |x_1| \leq \epsilon X} \bar{\zeta}_j(x) K\bar{\zeta}_j(x) dx \geq \alpha. \tag{2.44}$$

Henceforth, we assume that $0 < \lambda < \lambda_0$, $j \geq j_0$ and $\epsilon \geq \epsilon_0$. From the proof of Lemma 2.13 we may take

$$X = (\frac{2C\|\zeta_0\|_p}{\lambda})^{\frac{k}{2k-1}}.$$

Also for all $i \in \{0, 9\}$, we define the sequence $\{X_i\}_{i=0}^{i=9}$ as follows

$$X_i = \begin{cases} (\frac{2C\|\zeta_0\|_p}{\lambda})^{t_i} & \text{if } i \leq 8 \\ X & \text{if } i = 9, \end{cases}$$

where

$$t_i = t_0(2 - (\frac{1}{2})^i) \quad \text{and} \quad t_0 = \frac{2k}{(2k-1)(4-\tau)}.$$

Then it follows that

$$X_0 = X^{\frac{2}{4-\tau}}.$$

Now, let λ_1 be a positive number chosen so that $\epsilon_0 X \geq X_0$ for all $0 < \lambda < \lambda_1$. Since $\bar{\zeta}_j$ is Steiner-symmetric, then we have

$$\int_{x_2<X, |x_1|\leq \epsilon X} \bar{\zeta}_j(x) K\bar{\zeta}_j(x) dx \leq \frac{\epsilon X}{X_0} \int_{x_2<X, |x_1|\leq X_0} \bar{\zeta}_j(x) K\bar{\zeta}_j(x) dx$$

for all $\lambda \in (0, \lambda_1)$. Therefore from the form of X_0 and (2.44) we have

$$\begin{aligned}
\int_{x_2<X, |x_1|\leq X_0} \bar{\zeta}_j(x) K\bar{\zeta}_j(x) dx &\geq \frac{X_0}{\epsilon X} \int_{x_2<X, |x_1|\leq \epsilon X} \bar{\zeta}_j(x) K\bar{\zeta}_j(x) dx \\
&= \frac{X^{\frac{\tau-2}{4-\tau}}}{\epsilon} \int_{x_2<X, |x_1|\leq \epsilon X} \bar{\zeta}_j(x) K\bar{\zeta}_j(x) dx \\
&\geq \frac{\alpha}{\epsilon} X^{\frac{\tau-2}{4-\tau}}. \qquad (2.45)
\end{aligned}$$

Hence by Lemma 2.17 and (2.45) we find

$$\int_{|x_1|<X_0, x_2<X_0} x_2 \bar{\zeta}_j(x) dx + \sum_{i=1}^{i=9} \int_{|x_1|<X_0, X_{i-1}\leq x_2 < X_i} \bar{\zeta}_j(x) dx \geq \frac{\alpha}{\epsilon m} X^{\frac{\tau-2}{4-\tau}}, \qquad (2.46)$$

where

$$m = \|\zeta_0\|_p \max\{N, \widetilde{C} X^{1/k}\},$$

where $\widetilde{C}$ and N two positive constants depending on a and p. Since $\lambda \in (0, \lambda_0)$, then

$$m \leq \widetilde{C} \|\zeta_0\|_p X^{1/k}. \qquad (2.47)$$

Therefore by combining (2.46) and (2.47) with the form of X, we get

$$\int_{|x_1|<X_0, x_2<X_0} x_2 \bar{\zeta}_j(x) dx + \sum_{i=1}^{i=9} \int_{|x_1|<X_0, X_{i-1}\leq x_2 < X_i} \bar{\zeta}_j(x) dx \geq C(\alpha, \epsilon, k) \lambda^{\frac{k(2-\tau)+(4-\tau)}{(2k-1)(4-\tau)}}, \qquad (2.48)$$

where $C(\alpha, \epsilon, k) = \frac{\alpha}{\epsilon} (\frac{1}{2\widetilde{C}\|\zeta_0\|_p})^{\frac{(10-3\tau)k}{(2k-1)(4-\tau)}}$. We now set

$$C(\bar{\zeta}_j) = \int_{x_2<X} \frac{x_2}{1+|x|^2} \bar{\zeta}_j(x) dx.$$

Then we have

$$C(\bar{\zeta}_j) \geq \left(\int\limits_{|x_1|<X_0, x_2<X_0} + \sum_{i=1}^{i=9} \int\limits_{|x_1|<X_0, X_{i-1} \leq x_2 < X_i} \right) \frac{x_2}{1+|x|^2} \bar{\zeta}_j(x) dx. \tag{2.49}$$

If $|x_1| < X_0$ and $x_2 < X_0$, then

$$1 + |x|^2 < 3(\frac{2C\|\zeta_0\|_p}{\lambda})^{2t_0},$$

thus we have

$$\int\limits_{|x_1|<X_0, x_2<X_0} \frac{x_2}{1+|x|^2} \bar{\zeta}_j(x) dx \geq \frac{1}{3}(\frac{\lambda}{2C\|\zeta_0\|_p})^{2t_0} C_1(\bar{\zeta}_j), \tag{2.50}$$

where

$$C_1(\bar{\zeta}_j) = \int\limits_{|x_1|<X_0, x_2<X_0} x_2 \bar{\zeta}_j(x) dx.$$

Also if $|x_1| < X_0$ and $X_{i-1} \leq x_2 < X_i$, then

$$1 + |x|^2 \leq 3(\frac{2C\|\zeta_0\|_p}{\lambda})^{2t_i}.$$

Hence we find that

$$\int\limits_{|x_1|<X_0, X_{i-1} \leq x_2 < X_i} \frac{x_2}{1+|x|^2} \bar{\zeta}_j(x) dx \geq \frac{1}{3}(\frac{\lambda}{2C\|\zeta_0\|_p})^{2t_i - t_{i-1}} C_2(\bar{\zeta}_j), \tag{2.51}$$

where

$$C_2(\bar{\zeta}) = \int\limits_{|x_1<X_0, X_{i-1} \leq X_i} \bar{v}_j(x) dx.$$

Now for all $i \in \{0, 9\}$, we have

$$2t_0 = 2t_1 - t_0 = \cdots = 2t_i - t_{i-1} = \cdots = \frac{2k}{2k-1} - t_8, \tag{2.52}$$

hence it follows from (2.52) that

$$\int\limits_{|x_1|<X_0, X_i \leq x_2 < X_i} \frac{x_2}{1+|x|^2} \bar{\zeta}_j(x) dx \geq \frac{1}{3}(\frac{\lambda}{2C\|\zeta_0\|_p})^{2t_0} C_2(\bar{\zeta}_j). \tag{2.53}$$

Thus from (2.48), (2.49), (2.50), (2.51), and (2.53) we find

$$C(\bar{\zeta}_j) \;\; \geq \;\; \frac{1}{3}(\frac{\lambda}{2C\|\zeta_0\|_p})^{2t_0} \left(\int\limits_{|x_1|<X_0, x_2<X_0} x_2 \bar{\zeta}_j(x) dx + \sum_{i=1}^{i=9} \int\limits_{|x_1|<X_0, X_{i-1} \leq x_2 < X_i} \bar{\zeta}_j(x) dx \right)$$

$$\begin{aligned}
&\geq \frac{C(\alpha,\epsilon,k)}{3(2C\|\zeta_0\|_p)^{2t_0}}\lambda^{\frac{k(2-\tau)+(4-\tau)}{(2k-1)(4-\tau)}+2t_0}\\
&= \frac{C(\alpha,\epsilon,k)}{3(2C\|\zeta_0\|_p)^{2t_0}}\lambda^{\frac{(6-\tau)k+(4-\tau)}{(2k-1)(4-\tau)}}. \qquad (2.54)
\end{aligned}$$

Now let us set

$$A = \frac{C(\alpha,\epsilon,k)}{3(2C\|\zeta_0\|_p)^{2t_0}}.$$

Therefore using Lemma 2.18 and (2.54) we obtain

$$|\{x \in \Pi(\xi_j, X) | K\bar{\zeta}_j(x) - \frac{\lambda}{2}x_2^2 > 0\}| \geq G_k(\lambda),$$

where

$$G_k(\lambda) = \frac{\pi}{2}\left(\sqrt{\frac{A}{2\pi}}(\frac{1}{\lambda})^{\frac{(2-\tau)k-2(4-\tau)}{2(2k-1)(4-\tau)}} - 1\right)$$

provided $0 < \lambda < \min\{\lambda_0, \lambda_1\}$. Since A independent of λ and j, also $k > \frac{2(4-\tau)}{2-\tau}$, then it follows that $G_k(\lambda) \to \infty$ as $\lambda \to 0$. Hence we can choose $\delta > 0$ (independent of j) and λ_2 (independent of λ and j) such that

$$|\{\frac{A\lambda^{\frac{(6-\tau)k+(4-\tau)}{(2k-1)(4-\tau)}}x_2}{2\pi(1+|x|^2)} - \frac{\lambda}{2}x_2^2 \geq \delta\}| \geq \pi a^2.$$

This completes the proof. □

Chapter 3

A constrained variational problem for existence of a steady vortex on a planar domain

3.1 Introduction

Based on maximising the kinetic energy subject to the impulse being prescribed and the vorticity belonging to the set of rearrangements of a prescribed non-negative function ζ_0, Burton [13] proved the existence of steady flows in 2-dimensions containing symmetric vortex pairs. This variational principle is just an application of Benjamin's theory [5]. However, this Chapter deals with the existence theory of steady flows described by a variational problem, similar to the one governing steady 2-dimensional ideal fluid flows that contains symmetric vortex pairs. The approach which will be used here, is maximising the functional E, subject to vorticity ζ belonging to $\mathcal{W}(\zeta_0)$, the weak closer of the set $\mathcal{F}(\zeta_0)$, and subject to the impulse $I_n(\zeta) = I > 0$, where ζ_0 is a prescribed non-negative function and $n \geq 1$ is an integer number. Also in this Chapter, it will be shown that if I is large enough, and $\zeta_0 \in L^\infty$ having support of finite measure, then any maximiser of E subject to $\zeta \in \mathcal{W}(\zeta_0)$ and $I_n(\zeta) = I$, is in fact a rearrangement of ζ_0 for all $n \geq 1$.

To achieve the existence theorem for this variational problem, Burton's approach [18] for steady vortex rings with prescribed impulse has been followed. For specific values of n, this study may be compared to those of Burton [13] if $n = 1$, and to those of Burton and Emamizadeh [16] and Emamizadeh [22] for the case $n = 2$.

3.2 The main results

With the notions of Chapter 2, our main results are presented as follows:

Theorem 3.1. *Let $2 < p < \infty$ and let $\zeta_0 \in L^1(\Pi) \cap L^p(\Pi)$ be a non-negative function. Let $I > 0$ and let $n \geq 1$ be an integer number. Then the functional E attains a maximum value* *subject to $\zeta \in \mathcal{W}(\zeta_0)$ and $I_n(\zeta) = I$, and every maximiser is an element of $\mathcal{RC}(\zeta_0)$. For any maximiser ζ there exists an increasing function ϕ and a positive number λ such that the function $\psi := K\zeta$ satisfies*

$$-\Delta\psi = \phi \circ (\psi - \frac{\lambda}{n} x_2^n) \tag{3.1}$$

almost everywhere in Π.

Corollary 3.2. *Let $\zeta_0 \in L^\infty(\Pi)$ be a non-negative function having support of finite measure. Let ζ be a maximiser of E subject to $\mathcal{W}(\zeta_0)$ and $I_n(\zeta) = I$. Then we have*

(i) If $n = 1$ or $n = 2$, there exists $I_* > 0$ such that if $I \geq I_*$, then $\zeta \in \mathcal{F}(\zeta_0)$.

(ii) If $n \geq 3$, then $\zeta \in \mathcal{F}(\zeta_0)$ for all $I > 0$.

These results should be compared to the main result of Chapter 2 [Theorem 2.1], where we proved the existence of a maximiser for the functional $E - \lambda I_n$ relative to $\mathcal{F}(\zeta_0)$ for all $\lambda > 0$ if $n \geq 3$ and for small positive λ if $n = 2$. Note that in both Chapters, we construct solutions of the equation (3.1). Therefore, the cases of physical interest can be easily obtained.

3.3 Properties of K and the functional E

Let us start with some lemmas presenting some properties and estimates for the function $K\zeta$ and the functional E. Through this section and the next section, for all $X > 0$ we set $\Pi_X = \mathbb{R} \times (0, X)$.

Lemma 3.3. *Let $n \geq 1$, let $2 < p < \infty$ and let $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ be a non-negative function. Then there exists a positive number $\sigma_n(\zeta) = A_1(\|\zeta\|_1 + \|\zeta\|_p) + A_2 I_n(\zeta)$ such that for all $x_2 > 2$ we have*

$$K\zeta(x) \leq \frac{\sigma_n(\zeta)}{\sqrt{x_2}},$$

where A_1 and A_2 two positive constants.

Proof. For $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ and for all $x_2 > 2$ we set

$$F_1(x) = \int_{y_2 < x_2/2} G(x,y)\zeta(y)dy,$$

$$F_2(x) = \int_{y_2 \geq x_2/2, \rho \geq x_2^{-\alpha}} G(x,y)\zeta(y)dy$$

and

$$F_3(x) = \int_{y_2 \geq x_2/2, \rho < x_2^{-\alpha}} G(x,y)\zeta(y)dy,$$

where $\alpha = \frac{nq}{2-q}$ and $\rho = |x-y|$. For all $x \in \Pi$ and $y \in \Pi$ we have

$$G(x,y) = \frac{1}{4\pi} \log\left(1 + \frac{4x_2y_2}{|x-y|^2}\right) \le \frac{x_2y_2}{\pi|x-y|^2}. \tag{3.2}$$

Then applying (3.2) we find

$$\begin{aligned} F_1(x) &\le \frac{1}{\pi} \int\limits_{y_2<x_2/2} \frac{x_2y_2}{(x_2-y_2)^2}\zeta(y)dy \\ &\le \frac{4}{\pi x_2} I_1(\zeta) \\ &\le \frac{4}{\pi x_2}(\|\zeta\|_1 + nI_n(\zeta)). \end{aligned} \tag{3.3}$$

Now if $\rho \ge x_2^{-\alpha}$ and $x_2 > 2$, then it follows that

$$\frac{|x-\bar{y}|}{|x-y|} \le \frac{\rho + 2x_2}{\rho} \le x_2^{\alpha+3};$$

hence

$$G(x,y) = \frac{1}{2\pi}\log\left(\frac{|x-\bar{y}|}{|x-y|}\right) \le \frac{3+\alpha}{2\pi}\log x_2.$$

Therefore we have

$$\begin{aligned} F_2(x) &\le \frac{(\alpha+3)}{2\pi}\log x_2 \int\limits_{y_2\ge x_2/2} \zeta(y)dy \\ &\le \frac{2^{n-1}n(\alpha+3)\log x_2}{\pi x_2^n} I_n(\zeta). \end{aligned} \tag{3.4}$$

It remains just to find an estimate for F_3. Indeed, if $\rho < x_2^{-\alpha}$ and $x_2 > 2$, then

$$|x-\bar{y}| < \rho + 2x_2 < x_2^3.$$

Thus we have

$$\begin{aligned} F_3(x) &\le \frac{1}{2\pi} \int\limits_{y_2\ge x_2/2, \rho<x_2^{-\alpha}} (\log\frac{x_2^3}{\rho})\zeta(y)dy \\ &= \frac{3\log x_2}{2\pi} \int\limits_{y_2\ge x_2/2,\rho<x_2^{-\alpha}} \zeta(y)dy + \frac{1}{2\pi}\int\limits_{y_2\ge x_2/2,\rho<x_2^{-\alpha}} (\log\frac{1}{\rho})\zeta(y)dy \\ &\le \frac{3\log x_2}{2\pi}\int\limits_{y_2\ge x_2/2} \zeta(y)dy + \frac{1}{2\pi}\int\limits_{\rho<x_2^{-\alpha}} (\log\frac{1}{\rho})\zeta(y)dy. \end{aligned} \tag{3.5}$$

By using Hölder's inequality,

$$\int\limits_{\rho<x_2^{-\alpha}} (\log\frac{1}{\rho})\zeta(y)dy \le \left(2\pi\int_0^{x_2^{-\alpha}} (\log\frac{1}{\rho})^q \rho d\rho\right)^{1/q} \|\zeta\|_p.$$

Hence by making the changing of variable $u = \frac{1}{\rho}$, we find

$$\int_0^{x_2^{-\alpha}} (\log\frac{1}{\rho})^q \rho d\rho = \int\limits_{x_2^{\alpha}}^{\infty} \frac{(\log u)^q}{u^3} du \le \int\limits_{x_2^{\alpha}}^{\infty} \frac{du}{u^{3-q}} = \frac{1}{(2-q)x_2^{\alpha(2-q)}}. \tag{3.6}$$

Since $\alpha = \frac{nq}{2-q}$, then it follows from Hölder's inequality and (3.6) that

$$\int\limits_{\rho<x_2^{-\alpha}} (\log\frac{1}{\rho})\zeta(y)dy \le (\frac{2\pi}{2-q})^{1/q} \frac{\|\zeta\|_p}{x_2^n}. \tag{3.7}$$

From (3.5) we have

$$\frac{3\log x_2}{2\pi} \int\limits_{y_2 \ge x_2/2} \zeta(y)dy \le \frac{2^{n+1} n \log x_2}{\pi x_2^n} I_n(\zeta). \tag{3.8}$$

Therefore from (3.3), (3.4), (3.7) and (3.8) we can find two positive constants A_1 and A_2 depending on q and n such that

$$\sigma_n(\zeta) = A_1(\|\zeta\|_1 + \|\zeta\|_p) + A_2 I_n(\zeta),$$

and

$$K\zeta(x) \le \frac{\sigma_n(\zeta)}{\sqrt{x_2}}.$$

This completes the proof. □

Lemma 3.4. *Let $X > 0$ and let $\zeta \in L^\infty(\Pi_X)$ be a function that is Steiner-symmetric having support of finite measure. Then there exists a positive number C depending on X, $|\text{supp}\,\zeta|$ and $\|\zeta\|_\infty$ such that for all $|x_1| > 2$ we have*

$$K\zeta(x) \le C|x_1|^{-1}.$$

Proof. The proof of this Lemma is similar to the proof of Lemma 2.19, so we need just follow this proof. Indeed, for all $x \in \Pi_X$ we set

$$H_1(x) = \int\limits_{|x_1-y_1|<1} G(x,y)\zeta(y)dy,$$

$$H_2(x) = \int\limits_{1\leq|x_1-y_1|\leq\frac{1}{2}|x_1|} G(x,y)\zeta(y)dy$$

and

$$H_3(x) = \int\limits_{|x_1-y_1|>\frac{1}{2}|x_1|} G(x,y)\zeta(y)dy.$$

By (3.2), for all $x \in \Pi_X$ and $y \in \Pi_X$ we have

$$G(x,y) \leq \frac{X^2}{\pi|x-y|^2}.$$

Hence by using the same calculations as in the proof of Lemma 2.19, we find that

$$H_3(x) \leq \frac{4X^2\|\zeta\|_1}{\pi}|x_1|^{-1} \quad \text{and} \quad H_2(x) \leq \frac{2X^2\|\zeta\|_1}{\pi}|x_1|^{-1}. \tag{3.9}$$

It remains then only to consider $H_1(x)$. Indeed, by using Lemma 2.2, we get

$$H_1(x) \leq \frac{k(4X^2)^{1/k}}{2\pi} \int\limits_{|x_1-y_1|<1} \frac{\zeta(y)}{|x-y|^{2/k}}dy, \tag{3.10}$$

where $k > 2$. Since $|x-y|^{-2/k}$ is a decreasing rearrangement function with respect to $|x_2 - y_2|$, by applying then classical rearrangement inequality we have

$$\int\limits_{|x_1-y_1|<1} \frac{\zeta(y)}{|x-y|^{2/k}}dy \leq \int\limits_{|x_1-y_1|<1} \frac{\zeta_*(y)}{|x-y|^{2/k}}dy, \tag{3.11}$$

where $\zeta_*(y_1, .)$ is the rearrangement of $\zeta(y_1, .)$ that is symmetric decreasing about y_2, so if $y \in \text{supp } \zeta_*$, then it follows that $|x_2 - y_2| \leq M|y_1|^{-1}$, where $M = |\text{supp } \zeta|$. Now if $|x_1 - y_1| < 1$, then it follows that $|y_1| < 2|x_1|$ for all $|x_1| > 2$. Hence by using (3.11) we get

$$\begin{aligned}
\int\limits_{|x_1-y_1|<1} \frac{\zeta_*(y)}{|x-y|^{2/k}}dy &\leq \int\limits_{|x_1-y_1|<1,|x_2-y_2|<M|y_1|^{-1}} \frac{\zeta_*(y)}{|x-y|^{2/k}}dy \\
&\leq \int\limits_{|x_1-y_1|<1,|x_2-y_2|<2M|x_1|^{-1}} \frac{\zeta_*(y)}{|x-y|^{2/k}}dy \\
&\leq \|\zeta\|_\infty \int_{-2M|x_1|^{-1}}^{2M|x_1|} \frac{dsdt}{(t^2+s^2)^{1/k}} \\
&\leq 4M\|\zeta\|_\infty|x_1|^{-1}\int_{-1}^{1} \frac{dt}{|t|^{2/k}} \\
&= \frac{8kM\|\zeta\|_\infty}{k-2}|x_1|^{-1}.
\end{aligned} \tag{3.12}$$

Thus from (3.11) and (3.12) we find that

$$H_1(x) \leq \frac{2k^2(4X^2)^{1/k} M\|\zeta\|_\infty}{(k-2)\pi} |x_1|^{-1}. \tag{3.13}$$

Therefore by using the fact $\|\zeta\|_1 \leq M\|\zeta\|_\infty$ and $K\zeta(x) = \sum_{i=1}^{i=3} H_i(x)$, it follows then from (3.9) and (3.13) that there exists a positive number C depending on $\|\zeta\|_\infty$ and $|\text{supp}\ \zeta|$ such that

$$K\zeta(x) \leq C|x_1|^{-1}.$$

Hence the Lemma is proved. □

Lemma 3.5. *Let $1 < p < \infty$ and let $\zeta \in L^1(\Pi) \cap L^p(\Pi)$. For any $\beta > 0$, we define the function ζ_β by*

$$\zeta_\beta(x_1, x_2) = \begin{cases} \zeta(x_1, x_2 - \beta) & \textit{if } x_2 \geq \beta \\ 0 & \textit{if } x_2 < \beta. \end{cases}$$

Then for all $\beta > \alpha > 0$ we have

$$E(\zeta_\beta) - E(\zeta_\alpha) > \left(\frac{\sqrt{2}(\beta - \alpha)}{16\pi(1+\beta^2)} \right) \widetilde{E}(\zeta),$$

where

$$\widetilde{E}(\zeta) = \left(\int_\Pi \frac{\sqrt{x_2}\zeta(x)}{1+|x|^2} dx \right)^2 .$$

Proof. We have

$$\begin{aligned} E(\zeta_\beta) &= \frac{1}{2} \int_\Pi \zeta_\beta(x) K\zeta_\beta(x) dx \\ &= \frac{1}{8\pi} \int_\Pi \int_\Pi \log\left(\frac{(x_1-y_1)^2 + (x_2+y_2)^2}{(x_1-y_1)^2 + (x_2-y_2)^2} \right) \zeta(x_1, x_2-\beta)\zeta(y_1, y_2-\beta) dx dy. \end{aligned}$$

By making a change of variable, we get

$$E(\zeta_\beta) = \frac{1}{8\pi} \int_\Pi \int_\Pi \log\left(\frac{(x_1-y_1)^2 + (x_2+y_2+2\beta)^2}{(x_1-y_1)^2 + (x_2-y_2)^2} \right) \zeta(x)\zeta(y) dx dy.$$

Thus for $\beta > \alpha$

$$E(\zeta_\beta) - E(\zeta_\alpha) = \frac{1}{8\pi} \int_\Pi \int_\Pi \log\left(\frac{(x_1-y_1)^2 + (x_2+y_2+2\beta)^2}{(x_1-y_1)^2 + (x_2+y_2+2\alpha)^2} \right) \zeta(x)\zeta(y) dx dy. \tag{3.14}$$

Now observe that if $Z > Y > 0$, then

$$\log \frac{Z}{Y} > \frac{Z-Y}{Z}.$$

Hence it follows that

$$\begin{aligned}
\log\left(\frac{(x_1-y_1)^2+(x_2+y_2+2\beta)^2}{(x_1-y_1)^2+(x_2+y_2+2\alpha)^2}\right) &> \frac{(x_2+y_2+2\beta)^2-(x_2+y_2+2\alpha)^2}{(x_1-y_1)^2+(x_2+y_2+2\beta)^2} \\
&= \frac{4(\beta-\alpha)(x_2+y_2)+4(\beta^2-\alpha^2)}{(x_1-y_1)^2+(x_2+y_2+2\beta)^2} \\
&> \frac{4(\beta-\alpha)(x_2+y_2)}{(x_1-y_1)^2+(x_2+y_2+2\beta)^2}.
\end{aligned}$$

Since $(x_2+y_2)^2 \geq 2x_2y_2$, then we get

$$\log\left(\frac{(x_1-y_1)^2+(x_2+y_2+2\beta)^2}{(x_1-y_1)^2+(x_2+y_2+2\alpha)^2}\right) \geq \frac{4\sqrt{2}(\beta-\alpha)\sqrt{x_2y_2}}{(x_1-y_1)^2+(x_2+y_2+2\beta)^2}. \tag{3.15}$$

On the other hand we have

$$\begin{aligned}
(x_1-y_1)^2+(x_2+y_2+2\beta)^2 &\leq (x_1-y_1)^2+2(x_2+y_2)^2+8\beta^2 \\
&\leq 2((x_1-y_1)^2+(x_2+y_2)^2+4\beta^2) \\
&\leq 8(|x|^2+|y|^2+\beta^2) \\
&\leq 8(|x|^2+1)(|y|^2+1)(\beta^2+1).
\end{aligned} \tag{3.16}$$

Then from (3.15) and (3.16) we deduce that

$$\log\left(\frac{(x_1-y_1)^2+(x_2+y_2+2\beta)^2}{(x_1-y_1)^2+(x_2+y_2+2\alpha)^2}\right) > \frac{\sqrt{2}(\beta-\alpha)\sqrt{x_2y_2}}{2(\beta^2+1)(|x|^2+1)(|y|^2+1)}. \tag{3.17}$$

Therefore from (3.14) and (3.17), the result follows. □

Lemma 3.6. *Let $\zeta \in L^\infty(\Pi)$ be a function having bounded support and let $\psi = K\zeta$. Then we have*

$$\int_\Pi (x.\nabla\psi(x))\zeta(x)dx = 0.$$

Proof. The proof of this Lemma follows immediately from [13, Lemma 8]. □

Lemma 3.7. *Let $X > 0$ and let $\zeta \in L^\infty(\Pi_X)$ be a function having support of finite measure. Let $\psi = K\zeta$ and assume that ζ is Steiner symmetric. Then we have*

$$\int_{\Pi_X} (x.\nabla\psi(x))\zeta(x)dx = 0.$$

Proof. In the case when ζ has a bounded support, the proof follows from Lemma 3.6. Then we need just to consider the case when ζ has unbounded support of finite measure. To do this, let ζ_j be sequences defined by $\zeta_j = \zeta 1_{[-j,j]\times(0,X)}$ which converge to ζ strongly in L^p for all $1 \le p < \infty$; hence by using Lemma 2.4, we find that the sequence that defined by $\psi_j = K\zeta_j$ converge to ψ strongly in $L^\infty(\Pi_X)$. Now by using Lemma 3.6 we have

$$\int\limits_{\Pi_X} (x.\nabla\psi_j(x))\zeta_j(x)dx = 0.$$

Therefore it is sufficient just to show that

$$\int\limits_{\Pi_X} (x.\nabla\psi_j(x))\zeta_j(x) \to \int\limits_{\Pi_X} (x.\nabla\psi(x))\zeta(x)dx \quad \text{as} \quad j \to \infty.$$

Indeed, we set $\xi(x_2) = \sup\{x_1 | (x_1, x_2) \in \text{supp } \zeta\}$ and $\xi_j(x_2) = \min\{j, \xi(x_2)\}$. Then we have

$$\int\limits_{\Pi_X} x_1 \frac{\partial\psi_j}{\partial x_1}(\zeta_j(x) - \zeta(x))dx = \int_0^X \int\limits_{\xi_j(x_2)\le|x_1|<\xi(x_2)} x_1 \frac{\partial\psi_j}{\partial x_1}(\zeta_j(x) - \zeta(x))dx.$$

It follows that

$$\begin{aligned}
\left| \int\limits_{\Pi_X} x_1 \frac{\partial\psi_j}{\partial x_1}(\zeta_j(x) - \zeta(x))dx \right| &\le \|\zeta_j - \zeta\|_\infty \int_0^X \int\limits_{\xi_j(x_2)\le|x_1|<\xi(x_2)} |x_1|\left|\frac{\partial\psi_j(x)}{\partial x_1}\right|dx \\
&\le -4\|\zeta\|_\infty \int_0^X \int_{\xi_j(x_2)}^{\xi(x_2)} x_1 \frac{\partial\psi_j(x)}{\partial x_1}dx, \qquad (3.18)
\end{aligned}$$

because ψ_j is Steiner-symmetric. Applying now integration by parts, we get

$$\int_0^X \int_{\xi_j(x_2)}^{\xi(x_2)} x_1 \frac{\partial\psi_j(x)}{\partial x_1}dx = \int_0^X [x_1\psi_j(x)]_{\xi_j(x_2)}^{\xi(x_2)} dx_2 - \int_0^X \int_{\xi_j(x_2)}^{\xi(x_2)} \psi_j(x)dx. \qquad (3.19)$$

From Lemma 3.4, there exists a positive constant C depending only on X, $\|\zeta\|_\infty$ and $|\text{supp } \zeta|$ such that if $|x_1| > 2$, then $|x_1\psi_j(x)| \le C$. In other hand we have $\xi_j(x_2) = j$ if $\xi(x_2) > j$, and $\xi_j(x_2) = \xi(x_2)$ if $\xi(x_2) \le j$; hence by using (3.19) we get

$$\begin{aligned}
\int_0^X [x_1\psi_j(x)]_{\xi_j(x_2)}^{\xi(x_2)} dx_2 &\le \int_0^X (\xi(x_2) - \xi_j(x_2))\psi_j(\xi_j(x_2), x_2)dx_2 \\
&\le \frac{C_1}{j}|\text{supp } \zeta \backslash \text{supp } \zeta_j| \to 0 \quad \text{as} \quad j \to \infty, \qquad (3.20)
\end{aligned}$$

where C_1 is a positive number depending on C. Also by Lemma 3.4 we have

$$\int_0^X \int_{\xi_j(x_2)}^{\xi(x_2)} \psi_j(x)dx \le \frac{C}{j}|\text{supp } \zeta \backslash \text{supp } \zeta_j| \to 0 \quad \text{as} \quad j \to \infty. \qquad (3.21)$$

Thus from (3.18), (3.19), (3.20) and (3.21), for given $\varepsilon > 0$ we can choose $j_0 > 0$ such that for all $j \geq j_0$ we have

$$\left| \int\limits_{\Pi_X} x_1 \frac{\partial \psi_j}{\partial x_1} (\zeta_j(x) - \zeta(x)) dx \right| \leq \varepsilon/3. \tag{3.22}$$

Now for $t > 0$ we set $V(t) = \{y | \zeta(y) \geq t\}$ and $\xi^t(x_2) = \sup\{x_1 | \zeta(x_1, x_2) \geq t\}$. Since ζ is Steiner-symmetric, then using integration by parts yields

$$\begin{aligned}
\int\limits_{V(t)} x_1 \frac{\partial \psi_j(x)}{\partial x_1} dx &= 2 \int_0^X \int_0^{\xi^t(x_2)} x_1 \frac{\partial \psi_j(x)}{\partial x_1} dx \\
&= 2 \int_0^X \xi^t(x_2) \psi_j(\xi^t(x_2), x_2) dx_2 - 2 \int_0^X \int_0^{\xi^t(x_2)} \psi_j(x) dx. \quad (3.23)
\end{aligned}$$

Since $|\text{supp } \zeta| < \infty$, it follows that $\int_0^X \xi^t(x_2) dx_2 < \infty$; hence by Lemma 2.4, we deduce that there exists a positive number C_2 depending on X, $k > 1$, $p > 1$ and $|\text{supp } \zeta|$ such that

$$0 \leq \int_0^X \xi^t(x_2) \psi_j(\xi^t(x_2), x_2) dx_2 \leq C_2 \|\zeta\|_p$$

and

$$0 \leq \int_0^X \int_0^{\xi^t(x_2)} \psi_j(x) dx \leq C_2 \|\zeta\|_p.$$

Therefore we find

$$\left| \int\limits_{V(t)} x_1 \frac{\partial \psi_j(x)}{\partial x_1} dx \right| < 4C_2 \|\zeta\|_p. \tag{3.24}$$

Following the same calculations yields

$$\left| \int\limits_{V(t)} x_1 \frac{\partial \psi(x)}{\partial x_1} dx \right| < 4C_2 \|\zeta\|_p. \tag{3.25}$$

Combining (3.24) with (3.25) we obtain that

$$\int\limits_{V(t)} x_1 \left(\frac{\partial \psi_j(x)}{\partial x_1} - \frac{\partial \psi(x)}{\partial x_1} \right) dx$$

is absolutely convergent uniformly over t. Now by applying Fubini's theorem we have

$$\begin{aligned}
\int\limits_{\Pi_X} x_1 \left(\frac{\partial \psi_j(x)}{\partial x_1} - \frac{\partial \psi(x)}{\partial x_1} \right) \zeta(x) dx &= \int\limits_{\Pi_X} \int_0^{\zeta(x)} x_1 \left(\frac{\partial \psi_j(x)}{\partial x_1} - \frac{\partial \psi(x)}{\partial x_1} \right) dt dx \\
&= \int\limits_{\Pi_X} \int_0^{\|\zeta\|_\infty} 1_{V(t)}(x) x_1 \left(\frac{\partial \psi_n(x)}{\partial x_1} - \frac{\partial \psi(x)}{\partial x_1} \right) dt dx
\end{aligned}$$

$$= \int_0^{\|\zeta\|_\infty} \int_{\Pi_X} 1_{V(t)}(x) x_1 (\frac{\partial \psi_j(x)}{\partial x_1} - \frac{\partial \psi(x)}{\partial x_1}) dx dt$$

$$= \int_0^{\|\zeta\|_\infty} \int_{V(t)} x_1 (\frac{\partial \psi_j(x)}{\partial x_1} - \frac{\partial \psi(x)}{\partial x_1}) dx dt. \tag{3.26}$$

If now we want to prove that

$$\int_{\Pi_X} x_1 (\frac{\partial \psi_j(x)}{\partial x_1} - \frac{\partial \psi(x)}{\partial x_1}) \zeta(x) dx \to 0 \quad \text{as} \quad n \to \infty,$$

then from (3.26), it is sufficient in view of (3.24) and (3.25) just to show that

$$\int_{V(t)} x_1 (\frac{\partial \psi_j(x)}{\partial x_1} - \frac{\partial \psi(x)}{\partial x_1}) dx \to 0 \quad \text{as} \quad j \to \infty.$$

For that, we set

$$\int_{V(t)} x_1 (\frac{\partial \psi_j(x)}{\partial x_1} - \frac{\partial \psi(x)}{\partial x_1}) dx = 2 S_j(t)$$

where

$$S_j(t) = \int_0^X \int_0^{\xi^t(x_2)} x_1 (\frac{\partial \psi_j(x)}{\partial x_1} - \frac{\partial \psi(x)}{\partial x_1}) dx.$$

By using integration by parts again,

$$S_j(t) = \int_0^X \xi^t(x_2) \psi_j(\xi^t(x_2), x_2) - \psi(\xi^t(x_2), x_2)) dx_2 - \int_0^X \int_0^{\xi^t(x_2)} (\psi_j(x) - \psi(x)) dx. \tag{3.27}$$

Since $\psi_j \to \psi$ uniformly on Π_X, then by using Lebesgue's Dominated Convergence Theorem [27, Theorem D], we have

$$\int_0^X \int_0^{\xi^t(x_2)} (\psi_j(x) - \psi(x)) dx \to 0 \quad \text{as} \quad j \to \infty. \tag{3.28}$$

Also

$$\int_0^X \xi^t(x_2)(\psi_j(\xi^t(x_2), x_2) - \psi(\xi^t(x_2), x_2)) dx_2 \to 0 \quad \text{as} \quad j \to \infty. \tag{3.29}$$

Therefore from (3.26), (3.27), (3.28) and (3.29), for given $\varepsilon > 0$ we can choose $j_1 > 0$ such that for all $j \geq j_1$ we have

$$\left| \int_{\Pi_X} x_1 (\frac{\partial \psi_j(x)}{\partial x_1} - \frac{\partial \psi(x)}{\partial x_1}) \zeta(x) dx \right| < \varepsilon/3. \tag{3.30}$$

Now we write

$$\int\limits_{\Pi_X} x_1 \frac{\partial \psi_j}{\partial x_1}\zeta_j - \int\limits_{\Pi_X} x_1 \frac{\partial \psi}{\partial x_1}\zeta = \int\limits_{\Pi_X} x_1 (\frac{\partial \psi_j}{\partial x_1} - \frac{\partial \psi}{\partial x_1})\zeta + \int\limits_{\Pi_X} x_1 \frac{\partial \psi_j}{\partial x_1}(\zeta_j - \zeta).$$

Hence by using (3.22) and (3.30) we find

$$\left| \int\limits_{\Pi_X} x_1 \frac{\partial \psi_j(x)}{\partial x_1}\zeta_j(x) - \int\limits_{\Pi_X} x_1 \frac{\partial \psi(x)}{\partial x_1}\zeta(x) \right| \le \frac{2\varepsilon}{3} \tag{3.31}$$

for all $j \ge \max\{j_0, j_1\}$. It remains just to show that

$$\int\limits_{\Pi_X} x_2 \frac{\partial \psi_j(x)}{\partial x_2}\zeta_j(x)dx \to \int\limits_{\Pi_X} x_2 \frac{\partial \psi(x)}{\partial x_2}\zeta(x)dx \quad \text{as} \quad j \to \infty.$$

By Lemma 2.6, we can find a positive constant N independent of j such that $|\frac{\partial \psi_j(x)}{\partial x_2}| \le N$. Now since $\int\limits_{\Pi_X} x_2\zeta(x)dx < \infty$, then it follows from Lebesgue's Dominated Convergence Theorem [27, Theorem D] that

$$\int\limits_{\Pi_X} x_2 \frac{\partial \psi_j(x)}{\partial x_2}\zeta_j(x)dx \to \int\limits_{\Pi_X} x_2 \frac{\partial \psi(x)}{\partial x_2}\zeta(x)dx \quad \text{as} \quad j \to \infty.$$

Therefore given $\varepsilon > 0$, we can choose $j_2 > 0$ such that for all $j \ge j_2$ we have

$$\left| \int\limits_{\Pi_X} x_2 \frac{\partial \psi_j(x)}{\partial x_2}\zeta_j(x)dx - \int\limits_{\Pi_X} x_2 \frac{\partial \psi(x)}{\partial x_2}\zeta(x)dx \right| < \varepsilon/3. \tag{3.32}$$

By using the fact that $x.\nabla\psi(x) = x_2\frac{\partial \psi(x)}{\partial x_2} + x_1\frac{\partial \psi(x)}{\partial x_1}$, then it follows from (3.31) and (3.32), that for all $j \ge \max\{j_0, j_1, j_2\}$ we get

$$\left| \int\limits_{\Pi_X} (x.\nabla\psi_j(x))\zeta_j(x)dx - \int\limits_{\Pi_X} (x.\nabla\psi(x))\zeta(x)dx \right| < \varepsilon.$$

Therefore we deduce that

$$\int\limits_{\Pi_X} (x.\nabla\psi(x))\zeta(x)dx = 0.$$

This completes the proof. $\square$

Lemma 3.8. *Let the assumptions about X, Π_X and ζ be the same as in Lemma 3.7. Let λ be a non-negative number and let $\psi = K\zeta - \frac{\lambda}{n}x_2^n$, where $n \ge 1$. Suppose that $\zeta = \phi \circ \psi$ almost everywhere in Π_X for some increasing function ϕ, and suppose ϕ has a non-negative* *indefinite integral F. Then for all $\xi > 0$*

$$\int\limits_{\Pi(\xi,X)} (x.\nabla(\psi + \frac{\lambda}{n}x_2^n))\zeta(x)dx = -2 \int\limits_{\Pi(\xi,X)} F(\psi) + \lambda \int\limits_{\Pi(\xi,X)} x_2^n\zeta(x)dx + \int\limits_{\partial\Pi(\xi,X)} F(\psi)(x \cdot N)$$

where N is the outward unit normal, and consequently

$$\int\limits_{\Pi_X} F(\psi) \geq \frac{\lambda}{2} \int\limits_{\Pi_X} x_2^n\zeta(x)dx.$$

If additionally $F(s) = 0$ for $s \leq \beta$, then

$$\int\limits_{\Pi_X} \zeta(x)K\zeta(x)dx \geq \frac{(n+2)\lambda}{2n} \int\limits_{\Pi_X} x_2^n\zeta(x)dx + \beta\|\zeta\|_1.$$

Proof. The proof is an adaptation of the argument used in [13, Lemma 9] in the case when ζ has bounded support.

$$\int\limits_{\Pi(\xi X)} (x \cdot \nabla(\psi + \frac{\lambda}{n}x_2^n))\zeta(x)dx = \int\limits_{\Pi(\xi,X)} \left(x_1\frac{\partial\psi}{\partial x_1} + x_2\frac{\partial\psi}{\partial x_2}\right)\zeta(x) + \lambda \int\limits_{\Pi(\xi,X)} x_2^n\zeta(x)dx. \quad (3.33)$$

By using the fact that

$$\nabla(F \circ \psi) = (\phi \circ \psi)\nabla\psi,$$

then (3.33) becomes

$$\int\limits_{\Pi(\xi,X)} (x.\nabla(\psi + \frac{\lambda}{n}x_2^n))\zeta(x)dx = \int\limits_{\Pi(\xi,X)} x_1\frac{\partial}{\partial x_1}F(\psi) + x_2\frac{\partial}{\partial x_2}F(\psi) + \lambda \int\limits_{\Pi(\xi,X)} x_2^n\zeta(x)dx. \quad (3.34)$$

We can assume ϕ is bounded, hence F is Lipschitz, so $F \circ \psi(., x_2)$ is absolutely continuous for almost every x_2, so integration by parts relative to x_1 yields that

$$\int_{-\xi}^{\xi} x_1\frac{\partial}{\partial x_1}F(\psi)dx_1 = \xi F(\psi(\xi, x_2)) + \xi F(\psi(-\xi, x_2)) - \int_{-\xi}^{\xi} F(\psi)dx_1$$

for almost all x_2. Performing the same calculation for almost all x_1, we get

$$\int_0^X x_2\frac{\partial}{\partial x_2}F(\psi)dx_2 = XF(\psi(x_1, X)) - \int_0^X F(\psi)dx_2.$$

Substituting into (3.34), we find

$$\int\limits_{\Pi(\xi,X)} (x.\nabla(\psi + \frac{\lambda}{n}x_2^n))\zeta(x)dx = -2 \int\limits_{\Pi(\xi,X)} F(\psi) + \lambda \int\limits_{\Pi(\xi,X)} x_2^n\zeta(x)dx + \int\limits_{\partial\Pi(\xi,X)} F(\psi)(x \cdot N).$$

We now let $\xi \to \infty$; using Lemma 3.7 and the fact that $F \geq 0$ we obtain

$$2\int_{\Pi} F(\psi) \geq \lambda \int_{\Pi} x_2^n \zeta(x)dx. \tag{3.35}$$

As in [13, Lemma 9], we suppose $F(s) = 0$ for $s \leq \beta$, then $\phi(s) = 0$ for $s < \beta$ and ϕ is increasing. Hence we deduce that

$$F(s) = \int_{\beta}^{s} \phi \leq (s-\beta)\phi(s).$$

Therefore we have

$$\begin{aligned}
\int_{\Pi_X} \zeta(x)K\zeta(x)dx &= \int_{\Pi_X} \zeta(x)(\psi + \frac{\lambda}{n}x_2^n)dx \\
&= \int_{\Pi_X} \psi F'(\psi) + \frac{\lambda}{n}\int_{\Pi_X} x_2^n\zeta(x)dx \\
&= \int_{\Pi_X} (\psi-\beta)F'(\psi) + \beta\int_{\Pi_X} \zeta(x)dx + \frac{\lambda}{n}\int_{\Pi_X} x_2^n\zeta(x)dx \\
&\geq \int_{\Pi_X} F(\psi) + \beta\|\zeta\|_1 + \frac{\lambda}{n}\int_{\Pi_X} x_2^n\zeta(x)dx \\
&\geq \frac{(n+2)\lambda}{2n}\int_{\Pi_X} x_2^n\zeta(x)dx + \beta\|\zeta\|_1,
\end{aligned}$$

where we have used the inequality (3.35) to obtain the last line. □

Lemma 3.9. *Let X, n, λ, β and ζ satisfy the same assumptions as in Lemma 3.8. Then we have*

$$\max_{x\in\Pi_X}(K\zeta(x) - \frac{\lambda}{n}x_2^n) \geq \frac{2n}{(n+2)\|\zeta\|_1}E(\zeta) + \frac{2\beta}{n+2}.$$

Proof. We write

$$\int_{\Pi_X} \zeta K\zeta - \frac{\lambda}{n}\int_{\Pi_X} x_2^n\zeta = (\frac{n}{n+2})\int_{\Pi_X} \zeta K\zeta + \frac{2}{n+2}\left(\int_{\Pi_X} \zeta K\zeta - \frac{(n+2)}{2n}\lambda\int_{\Pi_X} x_2^n\zeta\right).$$

From Lemma 3.8, we have

$$\int_{\Pi_X} \zeta(x)K\zeta(x)dx \geq \frac{\lambda(n+2)}{2n}\int_{\Pi_X} x_2^n\zeta(x)dx + \beta\|\zeta\|_1.$$

Then it follows that

$$\int_{\Pi_X} \zeta(x)(K\zeta(x)dx - \frac{\lambda}{n}x_2^n)dx \geq \frac{2n}{n+2}E(\zeta) + \frac{2\|\zeta\|_1\beta}{n+2}.$$

Therefore we obtain

$$\max_{x\in\Pi_X}(K\zeta(x) - \frac{\lambda}{n}x_2^n) \geq \frac{2n}{(n+2)\|\zeta\|_1}E(\zeta) + \frac{2\beta}{(n+2)}.$$

This completes the proof. □

3.4 Proofs of the main results

In this section we will prove the main Theorem and Corollary, by following the same argument that Burton used in [13] and [18]. The proofs are broken into a series of lemmas, but first we recall some notation which will be used in the proofs of our results. For all $X > 0$ we let $\mathcal{W}_X(\zeta_0)$ denote the set of all functions ζ in $\mathcal{W}(\zeta_0)$ that are supported in Π_X, where $\zeta_0 \in L^1(\Pi) \cap L^p(\Pi)$ $(p > 1)$. For all $I > 0$ we set

$$\mathbb{T}(I) := \mathcal{W}_X(\zeta_0) \cap I_n^{-1}(I). \tag{3.36}$$

Also we define the function $\mathbb{F}_X : I_n(\mathcal{W}_X(\zeta_0)) \to \mathbb{R}_+$ by

$$\mathbb{F}_X(I) = \sup_{\mathbb{T}(I)} E(\zeta). \tag{3.37}$$

We recall the definition of the Hausdorff metric. If A and B two nonempty closed bounded sets in a metric space $(\mathbb{M}, d)$ we set $h(A,B) := \sup\{d(a,B) | a \in A\}$, where $d(a,B)$ denotes the usual distance from a to B, and we define the Hausdorff metric $d_H(A,B) := \max\{h(A,B), h(B,A)\}$. Finally, we denote by $\partial\mathbb{F}_X(I)$ the generalised gradient of $\mathbb{F}_X(I)$ defined in Clarke [19] at the point I as follows

$$\partial\mathbb{F}_X(I) = \left\{\widetilde{I} \in \mathbb{R}\, | \mathbb{F}_X^0(I;I^*) \geq \widetilde{I}I^* \quad \text{for all} \quad I^* \in \mathbb{R}\right\}, \tag{3.38}$$

where

$$\mathbb{F}_X^0(I;I^*) = \lim \sup_{\bar{I}\to I, t\downarrow 0} \frac{\mathbb{F}_X(\bar{I}+tI^*) - \mathbb{F}_X(\bar{I})}{t}. \tag{3.39}$$

Lemma 3.10. *Let $n \geq 1$, let $I > 0$, let $2 < p < \infty$ and let $\zeta_0 \in L^1(\Pi) \cap L^p(\Pi)$ be a non-negative function. Let $\mathcal{W}^s(\zeta_0)$ denote the set of all functions in $\mathcal{W}(\zeta_0)$ that are Steiner-symmetric about the x_2 axis. Let $\{\zeta_j\}_{j=1}^{\infty} \subset \mathcal{W}^s(\zeta_0)$ be a sequence converging to ζ in $L^p(\Pi)$ weakly such that $I_n(\zeta_j) \leq I$. Then we have*

$$E(\zeta_j) \to E(\zeta) \quad \text{as} \quad j \to \infty, \quad \text{and} \quad I_n(\zeta) \leq I.$$

Proof. From Lemma 3.3, we have

$$K\zeta(x) \leq \frac{\sigma_n(\zeta)}{\sqrt{x_2}}$$

for all $x_2 > 2$, where $\sigma_n(\zeta) = A_1(\|\zeta\|_1 + \|\zeta\|_p) + A_2 I_n(\zeta)$. Then it follows that, for any $\xi > 2$

$$\begin{aligned}\left|\int\limits_{x_2>\xi} (\zeta_j(x)K\zeta_j(x) - \zeta(x)K\zeta(x))dx\right| &\leq \int\limits_{x_2>\xi} \zeta_j(x)K\zeta_j(x)dx + \int\limits_{x_2>2} \zeta(x)K\zeta(x) \\ &\leq \frac{2A_1\|\zeta_0\|_1(\|\zeta_0\|_1 + \|\zeta_0\|_p) + 2A_2\|\zeta_0\|_1 I}{\sqrt{\xi}}.\end{aligned}$$

So for given $\varepsilon > 0$, we can find $\xi_0 > 2$ such that for all $\xi > \xi_0$ and all $j \in \mathbb{N}$

$$\left|\int\limits_{x_2>\xi} (\zeta_j(x)K\zeta_j(x) - \zeta(x)K\zeta(x))dx\right| < \frac{\varepsilon}{4}. \tag{3.40}$$

Since $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ is Steiner-symmetric, then by Lemma 2.5,

$$K\zeta(x) \leq M(\|\zeta_0\|_1 + \|\zeta_0\|_p)(x_2^{1/k} + x_2^{1/2k}) \min\{1, |x_1|^{\frac{-1}{2k+p}}\},$$

where M is a positive constant and $k \geq 1$. Hence for $|x_1| > \eta > 1$, we have

$$\begin{aligned}\left|\int\limits_{x_2<\xi_0,|x_1|>\eta} (\zeta_j(x)K\zeta_j(x) - \zeta(x)K\zeta(x))dx\right| &\leq \left|\int\limits_{x_2<\xi_0,|x_1|>\eta} \zeta_j(x)K\zeta_j(x)dx\right| \\ &\quad + \left|\int\limits_{x_2<\xi_0,|x_1|>\eta} \zeta(x)K\zeta(x)dx\right| \\ &\leq f(M, \xi_0, \eta),\end{aligned}$$

where

$$f(M, \xi_0, \eta) = 2M\|\zeta_0\|_1(\|\zeta_0\|_1 + \|\zeta_0\|_p)(\xi_0^{1/k} + \xi_0^{1/2k})\eta^{\frac{-1}{2k+p}}.$$

Thus for given $\varepsilon > 0$, we can choose $\eta_0 > 0$ such that for all $\eta > \eta_0$ and all $j \in \mathbb{N}$

$$\left|\int\limits_{x_2\leq\xi_0,|x_1|>\eta} (\zeta_j(x)K\zeta_j(x) - \zeta(x)K\zeta(x))dx\right| < \frac{\varepsilon}{4}. \tag{3.41}$$

It remains just to show that for given $\varepsilon > 0$ we have, for all large $j \in \mathbb{N}$

$$\left| \int_{\Pi(\xi_0,\eta_0)} (\zeta_j(x) K\zeta_j(x) - \zeta(x) K\zeta(x)) dx \right| < \frac{\varepsilon}{2}.$$

Indeed, we write

$$\begin{aligned} \int_{\Pi(\xi_0,\eta_0)} (\zeta_j(x) K\zeta_j(x) - \zeta(x) K\zeta(x)) dx &= \int_{\Pi(\xi_0,\eta_0)} (\zeta_j(x) - \zeta(x)) K\zeta(x) dx \\ &+ \int_{\Pi(\xi_0,\eta_0)} \zeta_j(x) (K\zeta_j(x) - K\zeta(x)) dx. \end{aligned}$$

By using Hölder's inequality we get

$$\left| \int_{\Pi(\xi_0,\eta_0)} \zeta_j(x) (K\zeta_j(x) - K\zeta(x)) dx \right| \le \|\zeta_0\|_p \|K\zeta_j - K\zeta\|_{L^q(\Pi(\xi_0,\eta_0))}.$$

From Lemma 2.7

$$K : L^p(\Pi(\xi_0,\eta_0)) \to L^q(\Pi(\xi_0,\eta_0))$$

is a compact operator, where q is the conjugate exponent of p. Thus by using [7, Remark 2, page 91] $K\zeta_j \to K\zeta$ strongly in $L^q(\Pi(\xi_0,\eta_0))$, so we can choose j_0 such that for all $j > j_0$

$$\left| \int_{\Pi(\xi_0,\eta_0)} \zeta_j(x) (K\zeta_j(x) - K\zeta(x)) dx \right| < \frac{\varepsilon}{4}. \tag{3.42}$$

Since $\{\zeta_j\}_{j \ge 1}$ converges weakly to ζ, then for given $\varepsilon > 0$ we can find j_1 such that

$$\left| \int_{\Pi(\xi_0,\eta_0)} (\zeta_j(x) - \zeta(x)) K\zeta(x) dx \right| < \frac{\varepsilon}{4} \tag{3.43}$$

for all $j > j_1$. Therefore it follows from (3.41), (3.42), (3.42) and (3.43) that

$$\left| \int_{\Pi} (\zeta_j(x) K\zeta_j(x) - \zeta(x) K\zeta(x)) dx \right| < \varepsilon$$

for all $j \ge \max\{j_0, j_1\}$. Hence we have

$$E(\zeta_j) \to E(\zeta) \quad \text{as} \quad j \to \infty.$$

Finally, I_n is lower-semi-continuous and convex on the non-negative function in L^p, and therefore weakly lower-semi-continuous, so $I_n(\zeta) \leq I$. $\square$

Lemma 3.11. *Let $2 < p < \infty$ and let α be a positive number. Then for all non-negative $\zeta \in L^1(\Pi) \cap L^p(\Pi)$ that satisfy $I_n(\zeta) < \infty$, we can find a positive number $X_0 > 2$ depending only on $\|\zeta\|_1$, $\|\zeta\|_p$ and $I_n(\zeta)$ such that if $E(\zeta) \geq \alpha$, then*

$$E(\zeta 1_{\Pi_{X_0}}) \geq \alpha/2.$$

Proof. For $X_0 > 0$ we set $\zeta_1 = \zeta 1_{\Pi_{X_0}}$ and $\zeta_2 = \zeta - \zeta_1 = \zeta 1_{\{x_2 > X_0\}}$. From Lemma 2.8, K is positive operator, and by using the fact that G is symmetric, then K is symmetric; hence we have

$$\begin{aligned} E(\zeta) &= \frac{1}{2}\int(\zeta_1(x)+\zeta_2(x))K(\zeta_1(x)+\zeta_2(x))dx \\ &= \frac{1}{2}\int \zeta_1(x)K\zeta_1(x)dx + \frac{1}{2}\int \zeta_1(x)K\zeta_2(x)dx + \frac{1}{2}\int \zeta_2(x)K\zeta(x)dx \\ &\leq E(\zeta_1) + \int \zeta_2(x)K\zeta(x)dx. \end{aligned}$$

It follows now from Lemma 3.3 that

$$K\zeta(x) \leq \frac{\sigma_n(\zeta)}{\sqrt{x_2}},$$

for all $x_2 > 2$. Hence we get

$$\begin{aligned} E(\zeta) &\leq E(\zeta_1) + \int\limits_{x_2 > X_0} \zeta_2(x)K\zeta(x)dx \\ &\leq E(\zeta_1) + \frac{\sigma_n(\zeta)\|\zeta\|_1}{\sqrt{X_0}}. \end{aligned}$$

Since $E(\zeta) \geq \alpha$, then we deduce that

$$E(\zeta_1) \geq \alpha - \frac{\sigma_n(\zeta)\|\zeta\|_1}{\sqrt{X_0}}.$$

Therefore if we take $X_0 = (\frac{2\sigma_n(\zeta)\|\zeta\|_1}{\alpha})^2$, we deduce that $E(\zeta_1) \geq \alpha/2$. Thus the result follows. $\square$

Lemma 3.12. *Let $k > 1$, let $\frac{k}{k-1} < p < \infty$ and let $\zeta_0 \in L^1(\Pi) \cap L^p(\Pi)$ be a non-negative function. Let $X > 0$ be given, and let α and ξ be two positive numbers. Then there exist two positive numbers β and ω such that for all $\zeta \in \mathcal{W}_X(\zeta_0)$ that satisfy $E(\zeta) \geq \alpha$, there is a square $S(\xi) \subset \Pi_X$ of side ξ for which we have*

$$|\{x \in S(\xi) | \zeta(x) \geq \omega\}| \geq \beta.$$

Proof. Let $1 < q < k$ be the conjugate exponent of p. We assume to seek a contradiction that for all positive ω and β, there exists $\zeta \in \mathcal{W}_X(\zeta_0)$ with $E(\zeta) \geq \alpha$, but for every square $S(\xi)$ of side ξ we have

$$|\{x \in S(\xi))|\zeta(x) \geq \omega\}| < \beta. \tag{3.44}$$

Consider fixed positive numbers ω and β, to be chosen later. Let $\zeta \in \mathcal{W}_X(\zeta_0)$ be such that $E(\zeta) \geq \alpha$, but for every square $S(\xi)$ we have (3.44). Let ρ denote the radius of the disc having area β, and for fixed $x \in \Pi_X$, we consider the square $S(N\xi)$ of side $N\xi$ and centre x, where N is a positive integer. Then we can cover $S(N\xi) \cap \Pi_X$ by a number N^2 of disjoint squares $\{C_j(\xi)\}_{j=1}^{N^2} \subset \Pi_X$ of side ξ. We recall now that for all x and y in Π_X, from Lemma 2.2 we have

$$G(x,y) \leq \frac{2^{1/k}k}{2\pi}\frac{(x_2y_2)^{1/2k}}{|x-y|^{1/k}} \leq \frac{2^{1/k}kX^{1/k}}{2\pi|x-y|^{1/k}}. \tag{3.45}$$

Hence by applying (3.45) we have

$$\begin{aligned}
K\zeta(x) &= \frac{1}{2\pi}\left(\int\limits_{S(N\xi)\cap\Pi_X} + \int\limits_{\Pi_X\backslash S(N\xi)}\right) G(x,y)\zeta(y)dy \\
&\leq \frac{2^{1/k}kX^{1/k}}{2\pi}\left(\int\limits_{S(N\xi)\cap\Pi_X} + \int\limits_{\Pi_X\backslash S(N\xi)}\right)\frac{\zeta(y)}{|x-y|^{1/k}}dy \\
&\leq \frac{2^{1/k}kX^{1/k}}{2\pi}\sum_{j=1}^{j=N^2}\int\limits_{C_j(\xi)}\frac{\zeta(y)}{|x-y|^{1/k}}dy + \frac{2^{1/k}kX^{1/k}}{2\pi}\int\limits_{|x-y|>\frac{N\xi}{2}}\frac{\zeta(y)}{|x-y|^{1/k}}dy \\
&\leq \frac{2^{1/k}kX^{1/k}}{2\pi}\sum_{j=1}^{j=N^2}\int\limits_{C_j(\xi)}\frac{\zeta(y)}{|x-y|^{1/k}}dy + \frac{2^{1/k}k(2X)^{1/k}\|\zeta_0\|_1}{2\pi(N\xi)^{1/k}}. \qquad (3.46)
\end{aligned}$$

We now write

$$\int\limits_{C_j(\xi)}\frac{\zeta(y)}{|x-y|^{1/k}}dy = \int\limits_{C_j(\xi),\zeta(y)\geq\omega}\frac{\zeta(y)}{|x-y|^{1/k}}dy + \int\limits_{C_j(\xi),\zeta(y)<\omega}\frac{\zeta(y)}{|x-y|^{1/k}}dy.$$

By using (3.44), the classical rearrangement inequality and Hölder's inequality we get

$$\begin{aligned}
\int\limits_{C_j(\xi),\zeta(y)\geq\omega}\frac{\zeta(y)}{|x-y|^{1/k}}dy &\leq \|v1_{C_j(\xi)}\|_p\left(\int\limits_{C_j(\xi)}\frac{dy}{|x-y|^{q/k}}\right)^{1/q} \\
&\leq \|\zeta\|_p\left(\int\limits_{|x-y|<\rho}\frac{dy}{|x-y|^{q/k}}\right)^{1/q} \\
&= \|\zeta\|_p(\frac{2k\pi}{2k-q})^{1/q}(\rho)^{\frac{2k-q}{kq}}. \qquad (3.47)
\end{aligned}$$

On the other hand we have

$$\int\limits_{C_j(\xi),\zeta(y)<\omega} \frac{\zeta(y)}{|x-y|^{1/k}}dy \le \omega \int\limits_{|x-y|<\frac{\xi}{\sqrt{\pi}}} \frac{dy}{|x-y|^{1/k}} = \frac{2k\pi}{2k-1}(\frac{\xi}{\sqrt{\pi}})^{\frac{2k-1}{k}}\omega. \tag{3.48}$$

Now from (3.47) and (3.48), we have

$$\int\limits_{C_j(\xi)} \frac{\zeta(y)}{|x-y|^{1/k}}dy \le \|\zeta\|_p(\frac{2k\pi}{2k-1})^{1/q}\rho^{\frac{2k-q}{kq}} + \frac{2k\pi}{2k-1}(\frac{\xi}{\sqrt{\pi}})^{\frac{2k-1}{k}}\omega.$$

Thus it follows from (3.46) that

$$K\zeta(x) \le C_1(X)(\rho^{\frac{2k-q}{2kq}} + (\frac{\xi}{\sqrt{\pi}})^{\frac{2k-1}{k}}\omega)N^2 + \frac{2^{1/k}k(2X)^{1/k}\|\zeta_0\|_1}{2\pi(N\xi)^{1/k}},$$

where $C_1(X) = \frac{2^{1/k}k}{2\pi}X^{1/k}(\frac{k\pi}{2k-q})^{1/q}\|\zeta\|_p$. Therefore we have

$$E(\zeta) \le C_1(X)\|\zeta_0\|_1(\rho^{\frac{2k-q}{kq}} + (\frac{\xi}{\sqrt{\pi}})^{\frac{2k-1}{k}}\omega)N^2 + \frac{2^{1/k}k(2X)^{1/k}\|\zeta_0\|_1^2}{2\pi(N\xi)^{1/k}}.$$

By choosing N large enough that $\frac{2^{1/k}k(2X)^{1/k}\|\zeta_0\|_1^2}{2\pi(N\xi)^{1/k}} < \alpha/2$, and also by choosing ω and ρ (and therefore β) small enough to ensure that $C_1(X)\|\zeta_0\|_1(\rho^{\frac{2k-q}{kq}} + (\frac{\xi}{\sqrt{\pi}})^{\frac{2k-1}{k}}\omega)N^2 < \alpha/2$, we find that $E(\zeta) < \alpha$. This a contradiction. Hence there exist two positive constants β and ω independent of ζ, such that if $\zeta \in \mathcal{W}_X(\zeta_0)$ satisfies $E(\zeta) \ge \alpha$ there is a square $S(\xi) \subset \Pi_X$ of side ξ such that

$$|\{x \in S(\xi)|\zeta(x) \ge \omega\}| \ge \beta.$$

This completes the proof. □

Lemma 3.13. *Let the assumptions about p, X, α, ξ and ζ_0 be the same as in Lemma 3.12. Then there exists a positive constant η such that for every $\zeta \in \mathcal{W}_X(\zeta_0)$ that satisfies $E(\zeta) \ge \alpha$, we have*

$$\int\limits_{S(\xi)} \sqrt{x_2}\zeta(x)dx \ge \eta,$$

where $S(\xi) \subset \Pi_X$ is some square of side ξ.

Proof. For given $\xi > 0$, there exist two positive constant ω and β such that if $\zeta \in \mathcal{W}_X(\zeta_0)$ satisfies $E(\zeta) \ge \alpha$, then there exists a square $S(\xi)$ of side ξ such that

$$|\{x \in S(\xi)|\zeta(x) \ge \omega\}| \ge \beta.$$

Now we consider $\zeta \in \mathcal{W}_X(\zeta_0)$ satisfying $E(\zeta) \ge \alpha$, we choose $S(\zeta)$ as just described, and we set

$$S = \{x \in S(\xi)|\zeta(x) \ge \omega\},$$

let δ be a positive number chosen so that $\delta\xi < \beta/2$, and let R be the rectangle defined by

$$R = [z_1 - \xi/2, z_1 + \xi/2] \times [z_2 - \xi/2 + \delta, z_2 + \xi/2] \subset \Pi_X,$$

where $z = (z_1, z_2)$ is the centre of $S(\xi)$. Then for all $x \in \Pi_X$ we have

$$\sqrt{x_2} 1_{S(\xi)}(x) \geq \sqrt{x_2} 1_D(y) \geq \sqrt{\delta} 1_D(x),$$

where $D = R \cap S$. Then

$$\int\limits_{S(\xi)} \sqrt{x_2}\zeta(x)dx \geq \int\limits_{S(\xi)} \sqrt{\delta} 1_D \zeta(x)dx = \sqrt{\delta} \int\limits_{D \cap S(\xi)} dx.$$

Notice that

$$\int\limits_D \zeta(x)dx \geq \omega \int\limits_D dx.$$

Since $|D| \geq \beta/2$, then it follows that

$$\int\limits_{S(\xi)} \sqrt{x_2}\zeta(x)dx \geq \eta,$$

where $\eta = \frac{\sqrt{\delta}\beta\omega}{2}$. □

Lemma 3.14. *Let $2 < p < \infty$, let $X > 0$ and let $\mathbb{B}_X = L^1(\Pi_X) \cap L^p(\Pi_X)$ be the Banach space associated with the norm $\|.\|_{\mathbb{B}_X} = \|.\|_1 + \|.\|_p$. Then $E \in C^1(\mathbb{B}_X)$.*

Proof. To prove this, let us consider ζ and h in $\mathbb{B}_X$. Since K is symmetric, then we have

$$E(\zeta + h) - E(\zeta) = E(h) + \int\limits_{\Pi_X} h(x)K\zeta(x)dx. \tag{3.49}$$

By using Lemma 2.5

$$E(h) \leq C\|h\|^2_{\mathbb{B}_X}(X^{1/2k} + X^{1/k}),$$

where $k \geq 1$ and C is a positive constant. Thus we obtain that

$$E(h) = o(\|h\|_{B_X}) \quad \text{as} \quad \|h\|_{\mathbb{B}_X} \to 0.$$

Therefore we deduce that E is Frechet differentiable. Now if we let E' denote its derivative, then from (3.49) we obtain

$$E'(\zeta)(h) = \int\limits_{\Pi_X} h(x)K\zeta(x)dx.$$

By Lemma 2.5 again, for $\zeta \neq \zeta'$ in $\mathbb{B}_X$ we have

$$|K(\zeta(x) - \zeta'(x))| \leq C\|\zeta - \zeta'\|_{\mathbb{B}_X}(X^{1/2k} + X^{1/k}).$$

Hence it follows that there exists a positive constant N depending on C and X such that

$$|(E'(\zeta) - E'(\zeta'))(h)| \leq N\|h\|_{\mathbb{B}_X}\|\zeta - \zeta'\|_{\mathbb{B}_X}.$$

Therefore $E'(\zeta)$ depends continuously on ζ. $\square$

Lemma 3.15. *$\mathbb{T}(.)$ is locally Lipschitz on $(0, \infty)$.*

Proof. In order to prove this Lemma, we should use the same procedure that Burton and Emamizadeh [16] used to prove Lemma 12; also we use [19, Definition 3.2]. Indeed, let $(a, b) \subset (0, \infty)$. Let I_1 and I_2 be two positive numbers in (a, b) with $I_1 > I_2$. Let us fix $\zeta_1 \in \mathcal{W}_X(\zeta_0) \cap I_n^{-1}(b)$. Then we can find a positive $\delta \in (0, 1)$ such that $I_1 = \delta b + (1 - \delta)I_2$; hence $\delta = \frac{I_1 - I_2}{b - I_2}$. Now for $\zeta_{I_2} \in \mathbb{T}(I_2)$ we have $I_n(\delta\zeta_1 + (1-\delta)\zeta_{I_2}) = \delta I_n(\zeta_1) + (1-\delta)I_n(\zeta_{I_2}) = I_1$. Thus $\delta\zeta_1 + (1 - \delta)\zeta_{I_2} \in \mathbb{T}(I_1)$. We deduce then that $\delta\zeta_1 + (1 - \delta)\mathbb{T}(I_2) \subseteq \mathbb{T}(I_1)$, or equivalently $\mathbb{T}(I_2) \subseteq \frac{-\delta}{1-\delta}\zeta_1 + \frac{1}{1-\delta}\mathbb{T}(I_1)$. Since the values of $\mathbb{T}(.)$ are convex, we can write $\mathbb{T} = \delta\mathbb{T} + (1 - \delta)\mathbb{T}$. Hence we find

$$\begin{aligned}
\mathbb{T}(I_2) &\subseteq \frac{-\delta}{1-\delta}\zeta_1 + \frac{\delta}{1-\delta}\mathbb{T}(I_1) + \mathbb{T}(I_1) \\
&\subseteq \frac{2\delta}{1-\delta}\|\zeta_0\|_{\mathbb{B}_X} B_1 + \mathbb{T}(I_1) \\
&\subseteq \frac{2(I_1 - I_2)}{b - I_1}\|\zeta_0\|_{\mathbb{B}_X} B_1 + \mathbb{T}(I_1), \qquad (3.50)
\end{aligned}$$

where B_1 is the unit ball in $\mathbb{B}_X = L^1(\Pi_X) \cap L^p(\Pi_X)$. A similar calculation shows that if $I_2 > I_1$, then we have

$$\mathbb{T}(I_1) \subseteq \frac{2(I_2 - I_1)}{I_2 - a}\|\zeta_0\|_p B_1 + \mathbb{T}(I_2). \qquad (3.51)$$

Hence it follows from (3.50) and (3.51) that there is a positive constant M depending on a and b such that

$$d_H(\mathbb{T}(I_1), \mathbb{T}(I_2)) \leq K|I_1 - I_2|,$$

provided I_1 and I_2 are bounded away from a and b. Therefore $\mathbb{T}(.)$ is locally Lipschitz on $(0, \infty)$. $\square$

Lemma 3.16. *The function $\mathbb{F}_X$ is locally Lipschitz on $(0, \infty)$.*

Proof. By Lemma 3.14 $E \in C^1(\mathbb{B}_X)$ and more strongly E is Lipschitz on the bounded set $\mathcal{W}_X(\zeta_0) \subset \mathbb{B}_X$ because E' is bounded on this set. Now let $[a, b] \subset (0, \infty)$ and consider I_1, I_2 in $[a, b]$; let us assume that $\mathbb{F}_X(I_1) \geq \mathbb{F}_X(I_2)$. Fix $\varepsilon > 0$, let $\zeta_1 \in \mathbb{T}(I_1)$ be chosen such that

$$\mathbb{F}_X(I_1) < E(\zeta_1) + \varepsilon,$$

then from (3.28) we have

$$\mathbb{F}_X(I_1) - \mathbb{F}_X(I_2) \leq E(\zeta_1) - E(\zeta) + \varepsilon, \quad \forall \zeta \in \mathbb{T}(I_2).$$

From the fact that E is Lipschitz on the bounded set $\mathcal{W}_X(\zeta_0) \subset \mathbb{B}_X$, we deduce that

$$\mathbb{F}_X(I_1) - \mathbb{F}_X(I_2) \leq k\|\zeta - \zeta_1\|_p + \varepsilon, \quad \forall \zeta \in \mathbb{T}(I_2),$$

where k is the Lipschitz constant of E. Using the definition of $d(\zeta_1, \mathbb{T}(I_2))$ yields the existence of $\zeta_2 \in \mathbb{T}(I_2)$ such that

$$\|\zeta_1 - \zeta_2\|_p \leq d(\zeta_1, \mathbb{T}(I_2)) + \varepsilon.$$

Thus

$$\mathbb{F}_X(I_1) - \mathbb{F}_X(I_2) \leq k(d(\zeta_1, \mathbb{T}(I_2)) + \varepsilon) + \varepsilon \leq k(d_H(\mathbb{T}(I_1), \mathbb{T}(I_2))) + (k+1)\varepsilon.$$

Using the same argument we find

$$\mathbb{F}_X(I_2) - \mathbb{F}_X(I_1) \leq k(d_H(\mathbb{T}(I_2), \mathbb{T}(I_1))) + (k+1)\varepsilon.$$

By letting $\varepsilon \to 0$, then we get

$$|\mathbb{F}_X(I_1) - \mathbb{F}_X(I_2)| \leq (k+1) d_H(\mathbb{T}(I_1), \mathbb{T}(I_2)).$$

Therefore by using Lemma 3.15, we deduce that $\mathbb{F}_X$ is locally Lipschitz. □

Lemma 3.17. *Let $I_0 < I_1$ be two positive numbers. Suppose that there exists $X > 0$ such that for all $I \in (I_0, I_1)$, there is a maximiser for $E(\zeta)$ subject to $\zeta \in \mathcal{W}(\zeta_0)$ and $I_n(\zeta) = I$, that is supported in Π_X. Then we have*

$$\lambda \in \partial \mathbb{F}_X(I) \Rightarrow \lambda > 0.$$

Proof. Consider $I \in (I_0, I_1)$ and for the given $X > 0$, let ζ be a maximiser for $E(w)$ subject to $w \in \mathcal{W}(\zeta_0)$ and $I_n(w) = I$, and supported in Π_X. We have

$$\mathbb{F}_X(I) = E(\zeta).$$

We need first to show that there exists a positive constant $\theta > 0$ such that

$$\mathbb{F}_X(I + h) \geq \mathbb{F}_X(I) + \theta h$$

for all small $h > 0$. To do that, let ζ_t denote the translation of ζ by $t \in (0,1)$ in the x_2 direction and take $h = I_n(\zeta_t) - I_n(\zeta)$. We recall that $n \geq 1$ is an integer number, so if

$n=1$ then it follows that $h = t\|\zeta\|_1$, and if $n>1$, then by Mean Value Theorem we have $h = tnI_{n-1}(\zeta_{\gamma t})$ for some $0<\gamma<1$; hence it follows that $h \leq 2^{n-1}t(2\|v\|_1 + I_1)$. Now using Lemma 3.5 yields

$$E(\zeta_t) \geq E(\zeta) + \frac{\sqrt{2}t}{16\pi(1+t^2)}\widetilde{E}(\zeta), \tag{3.52}$$

where

$$\widetilde{E}(\zeta) = \left(\int\limits_{\Pi_X} \frac{\sqrt{x_2}\zeta(x)}{1+|x|^2}dx \right)^2 .$$

Thus it suffices to find a positive constant C such that $\widetilde{E}(\zeta) \geq C$. Indeed, since ζ is a maximiser of E, then we can choose a positive number α depending on ζ_0, I_0 and I_1 only such that $E(\zeta) \geq \alpha$. Then Lemma 3.12 shows that there exist three positive numbers β, γ and ξ such that

$$|\{x \in S(\xi) | \zeta(x) \geq \gamma\}| \geq \beta,$$

where $S(\xi) \subset \Pi_X$ is some square of side ξ. Hence by Lemma 3.13, there exists a positive constant $\eta > 0$ depending on ζ_0, γ and β such that

$$\int\limits_{S(\xi)} \sqrt{x_2}\zeta(x)dx \geq \eta.$$

We can assume that ζ is Steiner symmetric, and then that $S(\xi)$ is Steiner symmetric. It follows then that

$$\int\limits_{S(\xi)} \frac{\sqrt{x_2}\zeta(x)}{1+|x|^2}dx \geq \frac{\eta}{1+(X+\frac{\xi}{2})^2} \geq \frac{4\eta}{9(1+X^2)}, \tag{3.53}$$

because $\xi < X$. Therefore we have

$$\widetilde{E}(\zeta) \geq \frac{16\eta^2}{81(1+X^2)^2}.$$

Thus by (3.52), (3.53) and the lower bound for t we deduce that there exists a positive constant θ such that for all $I_0 < I < I_1$ and $0 < h < I_1 - I$ we have

$$\mathbb{F}_X(I+h) - \mathbb{F}_X(I) \geq \theta h.$$

It then follows that the function $\mathbb{G}_X(I) = \mathbb{F}_X(I) - \theta I$ is increasing on $[I_0, I_1]$; hence by using the formula (3.39), we find that $\mathbb{G}_X^0(I;I^*) \geq 0$ for all $I^* > 0$ and $I_0 < I < I_1$. Therefore we get $\mathbb{G}_X^0(I,I^*) = C_1 I^*$, where $C_1 = \mathbb{G}_X^0(I,1) > 0$. Now if $I^* < 0$, then we can set $I^* = -I'$, where $I' > 0$; hence by using [19, Proposition 2.1.2] $\mathbb{G}_X^0(I,-I') = (-\mathbb{G}_X)^0(I,I') = C_2 I^*$, where $C_2 = (-\mathbb{G}_X)^0(I,1) \leq 0$ because $-\mathbb{G}_X$ is a decreasing function, so by using (3.38), we find that $\partial\mathbb{G}_X(I) = [-C_2, C_1]$, and therefore we deduce that $\forall\lambda \in \partial\mathbb{F}_X(I)$, we have $\lambda \geq \theta$. This completes the proof. $\square$

Proof of Theorem 3.1

For $I > 0$, we set

$$V(I) = \sup\{E(\zeta) | \zeta \in \mathcal{W}(\zeta_0), I_n(\zeta) \leq I\}. \tag{3.54}$$

Let $\mathcal{W}_X^s(\zeta_0)$ be the set of all functions in $\mathcal{W}_X(\zeta_0)$ that are Steiner-symmetric relative to the x_2 axis and supported in Π_X, where $\zeta_0 \in L^1(\Pi) \cap L^p(\Pi)$ $(p > 2)$. Let $\{\zeta_j\}_{j=1}^{\infty}$ be a maximising sequence of E relative to $\mathcal{W}(\zeta_0)$ with $I_n(\zeta_j) \leq I$, where $n \geq 1$ is an integer number. By using Lemma 2.9

$$E(\zeta_j^s) \geq E(\zeta_j)$$

and

$$I_n(\zeta_j^s) = I_n(\zeta_j) \leq I.$$

Hence ζ_j^s is a maximising sequence of E relative to $\mathcal{W}^s(\zeta_0)$ with $I_n(\zeta_j^s) \leq I$, and therefore by using Lemma 3.10 $E(\zeta)$ attains a maximum value relative to $\zeta \in \mathcal{W}^s(\zeta_0)$ and $I_n(\zeta) \leq I$. Then we need to show that every maximiser ζ must satisfy $I_n(\zeta) = I$. Indeed, let us consider $\bar{\zeta} \in \mathcal{W}^s(\zeta_0)$ a maximiser for (3.54) and assume that $I_n(\bar{\zeta}) < I$. For $\alpha > 0$, let $\bar{\zeta}_\alpha$ denote the translation of $\bar{\zeta}$ by $\alpha \in (0,1)$ in the x_2 direction; then using Lemma 3.5 yields

$$\begin{aligned} E(\bar{\zeta}_\alpha) &\geq E(\bar{\zeta}) + \frac{\sqrt{2}\alpha}{16\pi(1+\alpha^2)}\widetilde{E}(\bar{\zeta}) \\ &\geq E(\bar{\zeta}) + \frac{\sqrt{2}\alpha}{32\pi}\widetilde{E}(\bar{\zeta}). \end{aligned} \tag{3.55}$$

Also by making the change of variables $t_1 = x_1$ and $t_2 = x_2 - \alpha$ we have

$$\begin{aligned} I_n(\bar{\zeta}_\alpha) &= \frac{1}{n}\int x_2^n \bar{\zeta}(x_1, x_2 - \alpha)dx \\ &= \frac{1}{n}\int (t_2 + \alpha)^n \bar{\zeta}(t)dt \\ &= \frac{1}{n}\sum_{k=0}^{k=n} \binom{n}{k} \int t_2^k \alpha^{n-k} \bar{\zeta}(t)dt \\ &\leq I_n(\bar{\zeta}) + \alpha \sum_{k=0}^{k=n-1} \binom{n}{k} (\|\bar{\zeta}\|_1 + nI_n(\bar{\zeta})) \\ &= I_n(\bar{\zeta}) + \alpha(2^n - 1)(\|\bar{\zeta}\|_1 + nI_n(\bar{\zeta})). \end{aligned} \tag{3.56}$$

If

$$\alpha(2^n - 1)(\|\bar{\zeta}\|_1 + nI_n(\bar{\zeta})) < I - I_n(\bar{\zeta}),$$

then it follows that

$$I_n(\bar{\zeta}_\alpha) < I.$$

Hence if we choose $\alpha = \frac{1}{2}(\frac{I - I_n(\bar{\zeta})}{(2^n-1)(\|\bar{\zeta}\|_1 + nI_n(\bar{\zeta}))})$, then we find that

$$E(\bar{\zeta}_\alpha) > E(\bar{\zeta}) \quad \text{and} \quad I_n(\bar{\zeta}_\alpha) < I;$$

which shows us a contradiction. Therefore if $\bar{\zeta}$ is a maximiser subject to $\zeta \in \mathcal{W}(\zeta_0)$ and $I_n(\zeta) \leq I$, then $\bar{\zeta}$ must satisfy $I_n(\bar{\zeta}) = I$.

We are now going to show that for given $I_0 < I_1$, there exists a positive number X depending on I_0 and I_1 such that if $I_0 < I < I_1$ and $\bar{\zeta}$ is a maximiser for $E(\zeta)$ subject to $\zeta \in \mathcal{W}(\zeta_0)$ and $I_n(\zeta) = I$, then $\bar{\zeta}$ is supported in the region $\mathbb{R} \times (0, X)$. Indeed, consider $I_0 \leq I_1$ and assume that $\bar{\zeta}$ is a maximiser. Consider now $X > 0$ and let $w = \bar{\zeta}1_{\Pi_X}$, $h = \bar{\zeta} - w$, and assume that $h \neq 0$. We want to show that, for X large enough (depending only on I_1, $\|\zeta_0\|_1$ and $\|\zeta_0\|_p$), that this leads to a contradiction. Indeed, since G is symmetric, K is a symmetric operator. Also by Lemma 2.8, K is a positive operator, hence we have

$$\begin{aligned}
E(w) &= E(\bar{\zeta} - h) \\
&= E(\bar{\zeta}) - \frac{1}{2} \int\limits_{x_2 > X} \bar{\zeta}(x) K h(x) dx - \frac{1}{2} \int\limits_{x_2 > X} h(x) K \bar{\zeta}(x) dx \\
&\quad + \frac{1}{2} \int\limits_{x_2 > X} h(x) K h(x) dx \\
&= E(\bar{\zeta}) - \int\limits_{x_2 > X} h(x) K \bar{\zeta}(x) dx + \frac{1}{2} \int\limits_{x_2 > X} h(x) K h(x) dx \\
&\geq E(\bar{\zeta}) - \int\limits_{x_2 > X} h(x) K \bar{\zeta}(x) dx. \qquad (3.57)
\end{aligned}$$

Since $I_0 < I < I_1$, then it follows from Lemma 3.3 that

$$K\bar{\zeta}(x) \leq \frac{C(I_1)}{\sqrt{x_2}},$$

where

$$C(I_1) = A_1(\|\zeta_0\|_1 + \|\zeta_0\|_p) + A_2 I_1.$$

Also

$$\int\limits_{x_2 > X} h(x) dx \leq \frac{nI_n(h)}{X^n};$$

hence

$$\int\limits_{x_2 > X} h(x) K \bar{\zeta}(x) dx \leq \frac{C(I_1)}{X^{n+\frac{1}{2}}} I_n(h). \qquad (3.58)$$

Now for $\alpha > 0$, let w_α denote the translation of w by $\alpha \in (0,1)$ in the x_2 direction; then using (3.55) we get

$$E(w_\alpha) \geq E(w) + \frac{\sqrt{2}}{32\pi} \alpha \widetilde{E}(w).$$

Hence it follows from (3.57) and (3.58) that

$$E(w_\alpha) \geq E(\bar{\zeta}) + \frac{\sqrt{2}}{32\pi}\alpha\widetilde{E}(w) - \frac{C(I_1)}{X^{n+\frac{1}{2}}}I_n(h). \tag{3.59}$$

Also from (3.56) we have

$$\begin{aligned} I_n(w_\alpha) &\leq I_n(w) + \alpha(2^n - 1)(\|w\|_1 + nI_n(w)) \\ &\leq I_n(w) + \alpha(2^n - 1)(\|\zeta_0\|_1 + nI_1). \end{aligned} \tag{3.60}$$

Now if we want to have

$$I_n(w_\alpha) \leq I_n(\bar{\zeta}) = I,$$

then from (3.60) it will suffice to have

$$\alpha \leq \frac{I - I_n(w)}{(2^n - 1)(\|\zeta_0\|_1 + nI_1)} = \frac{I_n(h)}{(2^n - 1)(\|\zeta_0\|_1 + nI_1)}, \tag{3.61}$$

because $I_n(h) = I - I_n(w)$. Now if we want to have from (3.59) that $E(w_\alpha) > E(\bar{\zeta})$, then it is sufficient to have

$$\frac{\sqrt{2}}{32\pi}\alpha\widetilde{E}(w) - \frac{C(I_1)}{X^{n+\frac{1}{2}}}I_n(h) > 0. \tag{3.62}$$

Thus we need just find a positive constant e independent of $\bar{\zeta}$ such that $\widetilde{E}(w) \geq e$. For that, we recall that

$$\mathbb{F}_X(I) = \sup_{\mathbb{T}(I)} E(\zeta).$$

We note that for $I \geq I_0$ we have $\mathbb{F}_X(I) \geq \mathbb{F}_X(I_0)$. By using Lemma 3.11, we can choose $X_0 > 0$ such that if $\zeta \in \mathcal{W}(\zeta_0)$ and $E(\zeta) \geq m$; then we have $E(\zeta 1_{\mathbb{R}\times(0,X_0)}) \geq \frac{1}{2}m$, where $m = \mathbb{F}_X(I_0)$. Now we assume that $X \geq X_0$, then it follows from Lemma 3.12 that there exists a square $S(1) \subset \mathbb{R}\times(0, X_0)$ of side 1 and two positive constants γ and β (depending only on $\|\zeta_0\|_1$, $\|\zeta_0\|_p$ and I_1) such that

$$|\{x \in S(1) | w1_{\Pi_X}(x) \geq \beta\}| \geq \gamma.$$

Now Lemma 3.13 shows that there exists a positive constant η depending only on β and γ such that

$$\int_{S(1)} \sqrt{x_2}w(x)dx \geq \eta.$$

It follows then that

$$\int_{S(1)} \frac{\sqrt{x_2}}{1 + |x|^2}w(x)dx \geq e,$$

where $e = \frac{\eta}{1+(1+X_0+\frac{1}{2})^2}$, because w and $S(1)$ are Steiner-symmetric. Hence from (3.62) we need just to find X such that

$$\frac{\sqrt{2}}{32\pi}\alpha e^2 - \frac{C(I_1)}{X^{n+\frac{1}{2}}}I_n(h) > 0.$$

Now if we choose $\alpha = \frac{I_n(h)}{M(I_1)}$, where $M(I_1) = (2^n - 1)(\|\zeta_0\|_1 + nI_1)$, then we can choose $X > X_1 = (32\sqrt{2}\pi M(I_1)C(I_1)e^{-2})^{\frac{2}{2n+1}}$. Therefore if $X = \max\{X_0, X_1\}$ we find that $E(w_\alpha) > E(\bar{\zeta})$ which shows a contradiction. Hence there exists X large enough depending only on $\|\zeta_0\|_1$, $\|\zeta_0\|_p$ and I_1 such that for all $I \in [I_0, I_1]$ the maximiser $\bar{\zeta}$ is supported in the region $\mathbb{R} \times (0, X)$.

Henceforth for $I \in [I_0, I_1]$ we assume that $X = \max\{X_0, X_1\}$, and we recall the Banach space $\mathbb{B}_X = L^1(\Pi_X) \cap L^p(\Pi_X)$ $(p > 2)$ associated with the norm $\|.\|_{\mathbb{B}_X} = \|.\|_1 + \|.\|_p$. We also assume that $\bar{\zeta}$ is a maximiser for E relative to $\mathbb{T}(I)$. Since we do not know whether $\mathbb{F}_X$ is differentiable everywhere, we are going to use non-smooth analysis to obtain the Lagrange multiplier λ. By using Lemma 3.16, the function $\mathbb{F}_X$ is locally Lipschitz, and $\bar{\zeta}$ maximises $E - \mathbb{F}_X \circ I_n$ relative to $\mathcal{W}^s_X(\zeta_0) \cap \mathbb{B}_X$. Then $\bar{\zeta}$ maximises $E - \lambda I_n$ relative to $\mathcal{W}^s_X(\zeta_0)$, for some $\lambda \in \partial\mathbb{F}_X(I)$. Hence it follows from Lemma 3.17 that $\lambda > 0$. Therefore $\bar{\zeta}$ maximises

$$\int_\Pi \zeta(K\bar{\zeta} - \frac{\lambda}{n}x_2^n) \tag{3.63}$$

relative to $\mathcal{W}(\zeta_0) \cap \mathbb{B}_X$, where $\lambda > 0$. Since $-\Delta(K\bar{\zeta} - \frac{\lambda}{n}x_2^n) \geq \bar{\zeta}$ almost everywhere in Π_X, it follows from [11, Lemma 2.15] that there exists an increasing function ϕ such that $\bar{\zeta} = \phi \circ (K\bar{\zeta} - \frac{\lambda}{n}x_2^n)$ almost everywhere in Π_X. Since $\bar{\zeta}$ is a maximiser, then for every $Y \geq X$ we have $\bar{\zeta} = \phi_Y \circ (K\bar{\zeta} - \frac{\lambda}{n}x_2^n)$ on Π_Y, where ϕ_Y is increasing function. If $X \leq Y \leq Y_1$, then we can assume that ϕ_{Y_1} is an extension of ϕ_Y, hence we can choose an increasing function ϕ that is extension of all the ϕ_{Y_1}. Now if we set $\psi := K\bar{\zeta}$, then we have

$$-\Delta\psi = \phi \circ (\psi - \frac{\lambda}{n}x_2^n) \tag{3.64}$$

almost everywhere in Π for some increasing function ϕ and positive λ.

Now since $\bar{\zeta}$ maximises the strictly convex function E relative to the closed convex set $\mathcal{W}_X(\zeta_0) \cap I_n^{-1}(I) \cap L^1(\Pi_X)$, then $\bar{\zeta}$ is an extreme point of $\mathcal{W}_X(\zeta_0) \cap I_n^{-1}(I) \cap L^1(\Pi_X)$. Also since I_n is linear and bounded on $L^1(\Pi_X)$, then by applying [21, Lemma 2.4], we deduce $\bar{\zeta} \in \mathcal{RC}(\zeta_0)$.

Note that although Douglas's result is stated for $1 < p < \infty$, then the proof is valid when $p = 1$ also. □

Proof of Corollary 3.2

Let $0 < a < \infty$ and let $\zeta_0 \in L^\infty(\Pi)$ be a non-negative function having support of measure πa^2. Let $I > 0$, let $X > 0$ and $\bar{\zeta} \in \mathcal{W}^s_X(\zeta_0)$ be a maximiser for E relative to $\mathcal{W}_X(\zeta_0) \cap I_n^{-1}(I)$.

For all $n \geq 1$ we set

$$V(n) = \{x \in \Pi_X | K\bar{\zeta}(x) - \frac{\lambda}{n}x_2^n > 0\},$$

and

$$S = \{x|\bar{\zeta}(x) > 0\}.$$

The fact that $\bar{\zeta}$ is increasing function of $K\bar{\zeta}(x) - \frac{\lambda}{n}x_2^n$ on Π_X and (3.64) imply that apart from a set of zero measure

$$S \subset V(n).$$

To prove that $\bar{\zeta} \in \mathcal{F}(\zeta_0)$, we need first to show that if $|V(n)| \geq \pi a^2$, then $|S| \geq \pi a^2$. For that, we assume that $|V(n)| \geq \pi a^2$ but $|S| < \pi a^2$. Then there exists a rearrangement of $\zeta_0^\triangle - \bar{\zeta}^\triangle$ supported on the region $V(n)\backslash S$, because $\bar{\zeta} \in \mathcal{RC}(\zeta_0)$, where $\zeta_0^\triangle$ and $\bar{\zeta}^\triangle$ are decreasing rearrangements of ζ_0 and $\bar{\zeta}$ respectively. It follows then that $\bar{\zeta} + w$ is a rearrangement of ζ_0. Therefore we find that

$$\int (\bar{\zeta} + w)(K\bar{\zeta} - \frac{\lambda}{n}x_2^n) > \int \bar{\zeta}(K\bar{\zeta} - \frac{\lambda}{n}x_2^n)$$

which shows a contradiction with (3.63). It remains now just to show that $|V(n)| \geq \pi a^2$. Indeed, we begin with (ii). From Theorem 2.1, the set $V(n)$ has infinite measure for any $I > 0$ and for all $n \geq 3$. Thus we have $|V(n)| \geq \pi a^2$, and therefore $\bar{\zeta} \in \mathcal{F}(\zeta_0)$.

For (i), suppose that $n = 1$ or $n = 2$. From [13, Lemma 3], we can choose a number $N > 0$ such that

$$K\bar{\zeta}(x) \leq N x_2, \tag{3.65}$$

and by Lemma 2.6

$$|\nabla K\bar{\zeta}(x)| \leq N. \tag{3.66}$$

Since $\bar{\zeta} \in \mathcal{W}_X^s(\zeta_0)$ and $n \in \{1, 2\}$, then the set $V(n)$ does not have infinite measure if $n = 1$ or $n = 2$. Otherwise, we find that $\bar{\zeta} \in \mathcal{F}(\zeta_0)$ for any $I > 0$ and $n \geq 1$. We have, by definition,

$$V(n) \subset \Pi_X.$$

Let $z \in \Pi_X$ be such that for all $x \in \Pi_X$ we have

$$K\bar{\zeta}(z) - \frac{\lambda}{n}z_2^n \geq K\bar{\zeta}(x) - \frac{\lambda}{n}x_2^n.$$

From Lemma 3.9

$$K\bar{\zeta}(z) - \frac{\lambda}{n}z_2^n \geq \frac{2n}{(n+2)\|\zeta_0\|_1}E(\bar{\zeta}) + \frac{2\beta}{n+2}. \tag{3.67}$$

Since $\bar{\zeta}$ maximises the functional $\langle ., K\zeta - \frac{\lambda}{n}x_2^n\rangle$ on $\mathcal{W}_X(\zeta_0)$, so

$$S \subset V(n),$$

and from (3.64) we have

$$\bar{\zeta} = \phi \circ (K\bar{\zeta} - \frac{\lambda}{n}x_2^n).$$

Hence we can assume $\phi(t) = 0$ for $t \leq \beta$, where $\beta \geq 0$, and then it follows from (3.65) that

$$K\bar{\zeta}(z) - \frac{\lambda}{n}z_2^n \geq \frac{2n}{(n+2)\|\zeta_0\|_1}E(\bar{\zeta}).$$

By Lemma 2.10 we can choose $I_1 > 0$ such that for all $I \geq I_1$ we have $E(\bar{\zeta}) \geq C\log I$, where C is positive constant independent of I. Thus we get

$$K\bar{\zeta}(z) - \frac{\lambda}{n}z_2^n \geq \frac{2nC}{(n+2)\|\zeta_0\|_1}\log I.$$

Therefore it follows that

$$K\bar{\zeta}(z) - \frac{\lambda}{n}z_2^n \to \infty \quad \text{as} \quad I \to \infty;$$

hence there exists $I_* > 0$ such that for $I > \min\{I_*, I_1\}$ we have

$$K\bar{\zeta}(z) - \frac{\lambda}{n}z_2^n \geq 7aN.$$

Applying (3.65) we find

$$Nz_2 - \frac{\lambda}{n}z_2^n \geq 7aN.$$

Since $\lambda > 0$, the above inequality yields $z_2 > 7a$, and therefore Π_X contains at least a quadrant D of the half disc

$$\{x \in \Pi \,|\, |x - z| < 4a, x_2 < z_2\},$$

with $|D| = 4\pi a^2$. By using (3.66) and the Mean Value inequality, for all $x \in D$ and $I > \min\{I_*, I_1\}$ we have

$$\begin{aligned} K\bar{\zeta}(x) - \frac{\lambda}{n}x_2^n &\geq K\bar{\zeta}(z) - \frac{\lambda}{n}z_2^n - N|x - z| - \frac{\lambda}{n}(x_2^n - z_2^n) \\ &\geq K\bar{\zeta}(z) - \frac{\lambda}{n}z_2^n - 4aN \\ &\geq 3aN. \end{aligned}$$

Hence we obtain that the set defined by

$$\{x \in \Pi_X | K\bar{\zeta}(x) - \frac{\lambda}{n}x_2^n \geq aN\}$$

has measure greater that πa^2, for all $n \in \{1, 2\}$ and $I > \min\{I_1, I_*\}$. It follows then that for $I > \min\{I_1, I_*\}$, and $n \in \{1, 2\}$, $|V(n)| \geq \pi a^2$. Therefore we conclude that $\bar{\zeta} \in \mathcal{F}(v_0)$. This completes the proof. $\square$

3.5 Physical interpretation.

For $n \in \{1,2\}$, we set $\psi(x) = K\bar{\zeta}(x) - \frac{\lambda}{n}x_2^n$, where $\bar{\zeta}$ as in Corollary 3.2. Define Π_- to be the half-plane in $\mathbb{R}^2$ with $x_2 < 0$, and extend $K\bar{\zeta}$ from Π to $\mathbb{R}^2$ to be odd in x_2. According to our results, we have

$$-\Delta\psi(x) = \lambda(n-1)x_2^{n-2} + \phi_+(\psi(x)) \quad \text{if} \quad x \in \Pi, \tag{3.68}$$

where ϕ_+ is an increasing function. So if $x \in \Pi_-$, then this implies that $\bar{x} \in \Pi$; hence if $x \in \Pi_-$, then it follows that

$$-\Delta\psi(x) = -\Delta\psi(\bar{x}) = -\lambda(n-1)(-x_2)^{n-2} + \phi_+(\psi(\bar{x})) = (-1)^{n-1}\lambda(n-1)x_2^{n-2} - \phi_+(-\psi(x)).$$

Therefore we get

$$-\Delta\psi(x) = (-1)^{n-1}\lambda(n-1)x_2^n + \phi_-(\psi(x)) \quad \text{if} \quad x \in \Pi_-, \tag{3.69}$$

where $\phi_-(s) = -\phi_+(-s)$. Now if $x \in \partial\Pi$, then there exists a neighbourhood $\mathcal{N}$ of x such that ψ has the sign of x_2 in $\mathcal{N}$. Let $\delta = \max_{\Pi_X} \psi$ and define the function ϕ by $\phi(s) = \phi_+(s)$ if $s \in (0,\delta)$ and $\phi(s) = \phi_-(s)$ if $s \in (-\delta, 0)$; hence ϕ is an increasing function on $(-\delta,\delta)$. Thus if $x \in \mathcal{N}$ then if follows

$$-\Delta\psi(x) = \lambda(n-1)(-1)^{n-1}x_2^{n-2} + \phi(\psi(x)). \tag{3.70}$$

Therefore from (3.68), (3.69) and (3.70), we conclude that $-\Delta\psi$ is locally a function of ψ in $\mathbb{R}^2$. If $n=1$, then $\psi(x) = K\bar{\zeta}(x) - \lambda x_2$ is the stream function of steady symmetric vortex in a two-dimensional ideal fluid in an uniform flow. If $n=2$, then $\psi(x) = K\bar{\zeta}(x) - \frac{\lambda}{2}x_2^2$ is the stream function of steady vortex pairs in two "phase shear" flow. Henceforth, we assume that $n \in \{1,2\}$. The velocity field is given by

$$\left(\frac{\partial\psi(x)}{\partial x_2}, -\frac{\partial\psi(x)}{\partial x_1}\right) = \left(\frac{\partial K\bar{\zeta}(x)}{\partial x_2} - \lambda x_2^{n-1}, -\frac{\partial K\bar{\zeta}(x)}{\partial x_1}\right).$$

By Lemma 2.8 we have $K\bar{\zeta}(x) = O(|x|^{-1})$ and $|\nabla K\bar{\zeta}(x)| = O(|x|^{-2})$; so the flow at infinity approaches a stream function having velocity

$$(-\lambda x_2^{n-1}, 0).$$

Now the Euler equations for fluid flow are given as follow

$$(\mathrm{U}.\nabla)\mathrm{U} = -\nabla\mathrm{P} \quad \text{in} \quad D \tag{3.71}$$

$$\nabla.\mathrm{U} = 0, \quad \text{in} \quad D \tag{3.72}$$

where U is the velocity field, P is the pressure and $D \subset \mathbb{R}^2$ is a bounded domain symmetric in the x_2 axis. If we define $\mathrm{P}(x)$ by

$$\mathrm{P}(x) = \begin{cases} \frac{1}{2}|\nabla\psi|^2 + \Psi_+(\psi) & \text{for } x_2 > 0 \\ \frac{1}{2}|\nabla\psi|^2 + \Psi(\psi) & \text{for } -\delta < \psi < \delta, \\ \frac{1}{2}|\nabla\psi|^2 + \Psi_-(\psi) & \text{for } x_2 < 0, \end{cases}$$

where Ψ_+, Ψ_- and Ψ are the respective indefinite integrals of ϕ_+, ϕ_- and ϕ satisfying $\Psi_+(\delta) = \Psi_-(-\delta) = 0$ and $\Psi(0) = 0$. Then by calculating $(\mathrm{U}.\nabla)\mathrm{U}$, we find that ψ satisfies the equation (3.71). Therefore, we conclude that we have constructed the pressure P for which (3.68), (3.69), (3.70) and (3.72) hold almost everywhere.

Chapter 4

An existence theorem for steady vortex rings

4.1 Introduction

This Chapter contains a study of existence theory for a slightly different variational problem, governing a steady 3-dimensional ideal fluid flow containing axisymmetric steady vortex rings. For cylindrical coordinates (r, θ, z), we consider a flow whose Stokes stream function approaches at infinity $-\frac{\lambda}{2n}r^{2n}$, giving a velocity λr^{2n-1} in z-direction, where λ is a positive number and $n \geq 1$. The main result shows that for all λ positive and $n \geq 4$, a functional that is related to the kinetic energy has a maximiser belonging to $\mathcal{F}(v_0)$, the set of all rearrangements of a non-negative function v_0 which has support of finite volume. Additionally, if $2 \leq n < 4$, then under an assumption which we have not yet been able to justify, it will be shown that the same functional attains a maximum value relative to $\mathcal{F}(v_0)$ for all sufficiently small positive λ.

This existence theorem is therefore similar to the one that was investigated by Benjamin [5] and adapted by Burton [10]. The cases of physical relevance are $n = 1$ or $n = 2$.

By the symmetry in the r-axis, our problem can be reduced to one in the half-plane Π: $r > 0$, $-\infty < z < \infty$. Hence, the method that will be used to realise this existence theory, is similar to the method that used in Chapter 2. The variational principle suffers from the two difficulties of loss of compactness arising from Π, and the nature of the set $\mathcal{F}(v_0)$. To overcome these difficulties, the problem will first be solved on a bounded domain in Π by using Barton's results. The functional therefore has a maximiser in this bounded domain approximation. In the second step, the maximiser will be proved to be the same for all sufficiently large bounded domains. Thus the validity of the solution is established throughout the half-plane.

4.2 Mathematical formulation

For $\xi > 0$ and $X > 0$ we define the sets $\Pi(\xi, X) = \{(r,z) \in \Pi | r < X, |z| < \xi\}$. For $p \geq 1$, the definition of $L^p(\Pi, \nu)$ can be given as follows

$$L^p(\Pi, \nu) = \{v \text{ measurable on } \Pi \text{ and } \|v\|_p = (\int\limits_{\Pi} |v(r,z)|^p d\nu)^{1/p} < \infty\},$$

where ν denotes a measure on Π having density $2\pi r$ with respect to the 2-dimensional Lebesgue measure μ_2 on Π. Hence we use $\nu(A \cap \Pi)$ to denote the volume of any measurable cylindrically symmetric subset $A \subset \mathbb{R}^3$. The support of any function $v : \Pi \to \mathbb{R}$ is the set $\operatorname{supp} v = \{(r,z) \in \Pi | v(r,z) \neq 0\}$, thus if there exists $C > 0$ such that for all $(r,z) \in \Pi$ we have $r^2 + |z|^2 \leq C$, then the set $\operatorname{supp} v$ is bounded.

For $(r,z) \in \Pi$ and $(r', z') \in \Pi$, we set

$$G(r, r', z, z') = \frac{rr'}{8\pi^2} \int\limits_{-\pi}^{\pi} \frac{\cos\theta d\theta}{(r^2 + r'^2 - 2rr'\cos\theta + (z - z')^2)^{1/2}}; \tag{4.1}$$

then G is the Green's function for the operator $\mathcal{L}$ with homogeneous Dirichlet boundary conditions on Π, where the operator $\mathcal{L}$ is an elliptic partial differential operator given by

$$\mathcal{L} = -\left(\frac{1}{r}\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial z^2}\right).$$

For $v \in L^1(\Pi, \nu) \cap L^p(\Pi, \nu)$ $(p > 1)$ and for all $(r,z) \in \Pi$ we define the function Kv by

$$Kv(r,z) := \int\limits_{\Pi} G(r, r', z, z')v(r', z')d\nu'.$$

If $p > 5/2$, then Kv is the weak solution in the distribution sense for the problem $\mathcal{L}\Psi = v$ in Π, $\Psi(0,z) = 0$ and $\Psi(r,z) \to 0$ as $r^2 + z^2 \to \infty$. For all non-negative $v \in L^1(\Pi, \nu) \cap L^p(\Pi, \nu)$ $(p > 1)$ and $n \geq 1$, let $E(v)$ and $I_{2n}(v)$ be two functionals defined as in Chapter 1, section 4, then a linear combination of $E(v)$ and $I_{2n}(v)$ can be used to define the functional

$$\Phi_\lambda^n(v) = E(v) - \lambda I_{2n}(v),$$

where λ is a positive number.

Now with all these notations and formulations, we are able to present our main results as follows.

Theorem 4.1. *Let $n \geq 4$, let $5/2 < p < \infty$ and let $v_0 \in L^p(\Pi, \nu)$ be a non-negative function having support of finite volume. Let $\mathcal{F}(v_0)$ denote the set of rearrangements of v_0 on Π with respect to ν. Then for all $\lambda > 0$, the functional Φ_λ^n attains a maximum value relative to*

$\mathcal{F}(v_0)$. *Furthermore if v is a maximiser and $\Psi := Kv$, then we have*

$$\mathcal{L}\Psi = \phi \circ (\Psi - \frac{\lambda}{2n} r^{2n}) \tag{4.2}$$

almost everywhere in Π, for some increasing function ϕ.

Conjecture 4.2. *Let $2 < n \leq 4$ and let the assumptions about p, v_0 and $\mathcal{F}(v_0)$ be as in Theorem (4.1). Then there exists positive number λ_0 such that the functional Φ_λ^n attains a maximum value relative to $\mathcal{F}(v_0)$ for all $\lambda \in (0, \lambda_0)$. If v is a maximiser and $\Psi := Kv$, then Ψ satisfies (4.2) almost everywhere in Π.*

We will see that this conjecture can become a theorem under an assumption given as follows: If v is a maximiser of Φ_λ^n relative to $\mathcal{F}(\xi, X)$, the set of all rearrangements of v_0 on $\Pi(\xi, X)$ for various ξ and X depending on λ, then we have

$$\int_{r<X,|z|>X} v(r,z) Kv(r,z) d\nu \leq \beta,$$

where β is positive constant that may depend on X, in which case β does not tend to ∞ as X tends to ∞. We believe that this assumption can be realised by finding an estimate for the function Kv. In the case when $n = 1$ or $n = 2$, the maximiser v will be shown to give rise to a solution Ψ of the boundary value problem for axisymmetric steady vortex rings in uniform flow if $n = 1$, and Poiseuille flow if $n = 2$; hence Ψ is a function of (r, z) only in cylindrical coordinates, and satisfies

$$(P) \begin{cases} \mathcal{L}\Psi = \phi \circ (\Psi - \frac{\lambda}{2n} r^{2n}) & \text{in } \Pi, \\ \Psi(r,z) = 0 & \text{when } r = 0, \\ \Psi(r,z) \to 0 & \text{as } r^2 + z^2 \to \infty. \end{cases}$$

In the case $n = 1$ or $n = 2$, Ψ solves the above problem, then $\Psi - \frac{\lambda}{2n} r^{2n}$ is the Stokes stream function of a steady ideal fluid flow, whose velocity in $\mathbb{R}^3$ is given by

$$\left(-\frac{1}{r}\frac{\partial \Psi}{\partial z}, 0, \frac{1}{r}\frac{\partial \Psi}{\partial r} - \lambda r^{2n-2} \right).$$

4.3 Estimates and properties for the function Kv

We adopt the same strategy that has been used in Chapter 2. First of all, in this section we start with some Lemmas concerning the Green's function G, which allow us to find some properties and estimates for the function Kv.

Lemma 4.3. *Let (r, z) and (r', z') be points in Π, let $R = ((r - r')^2 + (z - z'))^{1/2}$ and*

$\overline{R} = ((r+r')^2 + (z-z')^2)^{1/2}$. *Then*

$$\frac{(rr')^2}{6\pi^2 \overline{R}^3} < G(r,r',z,z') \le \frac{rr'}{2\pi^2 \overline{R}} \log(\frac{9\overline{R}}{R}).$$

Proof. Since the integrand in (4.1) is even in θ, then by setting $a = r^2 + r'^2 + (z-z')^2$ and $b = rr'$, we have

$$G(r,r',z,z') = \frac{b}{4\pi^2} \int_0^\pi \frac{\cos\theta d\theta}{(a-2b\cos\theta)^{1/2}} \le \frac{b}{4\pi^2} \int_0^\pi \frac{d\theta}{(a-2b\cos\theta)^{1/2}}. \tag{4.3}$$

If we set $\tan\frac{\theta}{2} = u$, then $\cos\theta = \frac{1-u^2}{1+u^2}$. Hence from (4.3) we find

$$G(r,r',z,z') \le \frac{b}{2\pi^2} \int_0^\infty \frac{du}{\sqrt{(a-2b) + (a+2b)u^2)}\sqrt{1+u^2}}. \tag{4.4}$$

Since $R^2 = a - 2b$ and $\overline{R}^2 = a + 2b$, then it follows from (4.4) that

$$\begin{aligned} G(r,r',z,z') &\le \frac{b}{2\pi^2} \int_0^\infty \frac{du}{\sqrt{R^2 + (\overline{R}u)^2}\sqrt{1+u^2}} \\ &= \frac{b}{2\pi^2} \left(\int_0^1 + \int_1^\infty \right) \frac{du}{\sqrt{R^2 + (\overline{R}u)^2}\sqrt{1+u^2}}. \end{aligned} \tag{4.5}$$

Now by setting $\frac{\overline{R}}{R}u = t$ and using the fact that $\overline{R} > R$, we get

$$\begin{aligned} \int_0^1 \frac{du}{\sqrt{R^2 + (\overline{R}u)^2}\sqrt{1+u^2}} &\le \int_0^1 \frac{du}{\sqrt{R^2 + (\overline{R}u)^2}} \\ &= \frac{1}{\overline{R}} \int_0^{\frac{\overline{R}}{R}} \frac{dt}{\sqrt{1+t^2}} \\ &= \frac{1}{\overline{R}} \log(\sqrt{1 + (\frac{\overline{R}}{R})^2} + \frac{\overline{R}}{R}). \end{aligned} \tag{4.6}$$

If we set also $u = \frac{1}{t}$, then we have

$$\begin{aligned} \int_1^\infty \frac{du}{\sqrt{R^2 + (\overline{R}u)^2}\sqrt{1+u^2}} &= \int_0^1 \frac{dt}{\sqrt{\overline{R}^2 + (Rt)^2}\sqrt{1+t^2}} \\ &\le \frac{1}{\overline{R}} \int_0^1 \frac{dt}{\sqrt{1+t^2}} \le \frac{\log 3}{\overline{R}}. \end{aligned} \tag{4.7}$$

Thus it follows from (4.5), (4.6) and (4.7) that

$$G(r,r',z,z') \leq \frac{b}{2\pi^2\overline{R}} \log 3(\sqrt{1+(\frac{\overline{R}}{R})^2}+\frac{\overline{R}}{R}).$$

Since $\bar{R} > R$, then we find

$$G(r,r',z,z') \leq \frac{b}{2\pi^2\overline{R}} \log(\frac{9\overline{R}}{R}). \tag{4.8}$$

To obtain the lower bound for $G(r,r',z,z')$, we set $u = \cos\theta$. Then from (4.3) we find that

$$\begin{aligned}
G(r,r',z,z') &= \frac{b}{4\pi^2}\int_0^\pi \frac{\cos\theta d\theta}{(a-2b\cos\theta)^{1/2}} \\
&= \frac{b}{4\pi^2}\int_{-1}^1 \frac{udu}{(a-2bu)^{1/2}(1-u^2)^{1/2}} \\
&= \frac{b}{4\pi^2}\int_0^1 \frac{udu}{(a-2bu)^{1/2}(1-u^2)^{1/2}} - \frac{b}{4\pi^2}\int_0^{-1} \frac{udu}{(a-2bu)^{1/2}(1-u^2)^{1/2}} \\
&= \frac{b}{4\pi^2}\int_0^1 \frac{udu}{(a-2bu)^{1/2}(1-u^2)^{1/2}} - \frac{b}{4\pi^2}\int_0^{1} \frac{udu}{(a+2bu)^{1/2}(1-u^2)^{1/2}} \\
&= \frac{b}{4\pi^2}\int_0^1 \left(\frac{(a+2bu)^{1/2}-(a-2bu)^{1/2}}{(a+2bu)^{1/2}(a-2bu)^{1/2}}\right)\frac{udu}{(1-u^2)^{1/2}}.
\end{aligned}$$

Now since

$$(a+2bu)^{1/2}-(a-2bu)^{1/2} = \frac{4bu}{(a+2bu)^{1/2}+(a-2bu)^{1/2}} \quad \text{and} \quad a+2bu > a-2bu,$$

then it follows from above that

$$\begin{aligned}
G(r,r',z,z') &\geq \frac{b^2}{2\pi^2(a+2b)^{3/2}}\int_0^1 \frac{u^2du}{\sqrt{1-u^2}} \\
&> \frac{b^2}{6\pi^2(a+2b)^{3/2}} \\
&= \frac{(rr')^2}{6\pi^2((r+r')^2+(z-z')^2)^{3/2}}.
\end{aligned} \tag{4.9}$$

Therefore the result follows from (4.8) and (4.9). □

Lemma 4.4. *For any non-negative $v \in L^1(\Pi,\nu)\cap L^p(\Pi,\nu)$ $(p>1)$, and for all $(r,z)\in\Pi$*

we have

$$Kv(r,z) \geq \frac{r^2}{24\pi^2(1+|\rho|^3)} \int\limits_{\Pi} \frac{r'^2}{(1+|\rho'|^3)} v(r',z')d\nu',$$

where $\rho = (r,z)$ *and* $\rho' = (r',z')$. *Furthermore*

$$E(v) \geq \frac{1}{48\pi^2} \left(\int\limits_{\Pi} \frac{r^2}{1+\rho^3} v(r,z)d\nu \right)^2.$$

Proof. We use the same argument that has been used in order to prove Lemma 2.3. Indeed, since v is non-negative then by using Lemma 4.3, for all $\rho = (r,z) \in \Pi$ we have

$$Kv(r,z) \geq \frac{1}{6\pi^2} \int\limits_{\Pi} \frac{(rr')^2}{((r+r')^2+(z-z')^2)^{3/2}} v(r',z')d\nu'.$$

By setting $\bar{\rho} = (-r,z)$, we have

$$\begin{aligned} ((r+r')^2+(z-z')^2)^{3/2} &= |\bar{\rho}-\rho'|^3 \\ &\leq (|\rho|+|\rho'|)^3 \\ &\leq 4|\rho|^3+4|\rho'|^3 \\ &< 4(1+|\rho|^3)(1+|\rho'|^3). \end{aligned} \tag{4.10}$$

Thus

$$Kv(r,z) \geq \frac{r^2}{24\pi^2(1+|\rho|^3)} \int\limits_{\Pi} \frac{r'^2}{(1+|\rho'|^3)} v(r',z')d\nu'.$$

By following the same method that was used in Lemma 2.3, using the definition of E and (4.10), we find

$$\begin{aligned} E(v) &\geq \frac{1}{48\pi^2} \int\limits_{\Pi} \frac{r^2}{1+|\rho|^3} v(r,z)d\nu \int\limits_{\Pi} \frac{r'^2}{1+|\rho'|^3} v(r',z')d\nu' \\ &= \frac{1}{48\pi^2} \left(\int\limits_{\Pi} \frac{r^2}{1+|\rho|^3} v(r,z)d\nu \right)^2. \end{aligned}$$

Hence the result follows. □

Lemma 4.5. *Let* $k \geq 1$, *let* $\frac{2k}{2k-1} < p < \infty$, *let* q *the exponent conjugate of* p *and let* $v \in L^1(\Pi,\nu) \cap L^p(\Pi,\nu)$ *be a non-negative function. Then for all* $(r,z) \in \Pi$, *we have*

$$Kv(r,z) \leq M(\|v\|_1 + \|v\|_p)(r^{1+\frac{1}{2k}-\frac{1}{q}} + r^{1+\frac{1}{k}}),$$

where M *is a positive number depending on* k *and* q.

Proof. For all $(r,z) \in \Pi$ and $(r',z') \in \Pi$, we set $R^2 = (r-r')^2+(z-z')^2$ and $\overline{R}^2 =$

$(r+r')^2+(z-z')^2$. By Lemma 4.3 we have

$$Kv(r,z) \le \frac{1}{2\pi^2}\int\limits_{\Pi} \frac{rr'}{\overline{R}}\log(\frac{9\overline{R}}{R})v(r',z')d\nu' \le \frac{r}{2\pi^2}\int\limits_{\Pi}\log(\frac{9\overline{R}}{R})v(r',z')d\nu', \tag{4.11}$$

because $\overline{R} \ge r'$. By using Lemma 2.2 we have

$$\log(\frac{\overline{R}}{R}) \le 2^{1/k}k\frac{(rr')^{1/2k}}{R^{1/k}}. \tag{4.12}$$

Combining (4.11) with (4.12) we find

$$\begin{aligned} Kv(r,z) &\le \frac{kr}{\pi^2}\int\limits_{\Pi}\frac{(rr')^{1/2k}}{R^{1/k}}v(r',z')d\nu' + \frac{\log 9}{\pi^2}\|v\|_1 r \\ &\le \frac{kr}{\pi^2}\int\limits_{\Pi}\frac{(rR+r^2)^{1/2k}}{R^{1/k}}v(r',z')d\nu' + \frac{\log 9}{\pi^2}\|v\|_1 r \\ &\le \frac{k}{\pi^2}\left(r^{1+\frac{1}{2k}}\int\limits_{\Pi}\frac{v(r',z')}{R^{1/2k}}d\nu' + r^{1+\frac{1}{k}}\int\limits_{\Pi}\frac{v(r',z')}{R^{1/k}}d\nu'\right) + \frac{\log 9}{\pi^2}\|v\|_1 r. \end{aligned} \tag{4.13}$$

We now set

$$G_1(r,z) = \int\limits_{\Pi}\frac{v(r',z')}{R^{1/2k}}d\nu' \quad \text{and} \quad G_2(r,z) = \int\limits_{\Pi}\frac{v(r',z')}{R^{1/k}}d\nu'.$$

For G_1 we have

$$G_1(r,z) = (\int\limits_{R<r^{-1}} + \int\limits_{R\ge r^{-1}})\frac{v(r',z')}{R^{1/2k}}d\nu' \le \int\limits_{R<r^{-1}}\frac{v(r',z')}{R^{1/2k}}d\nu' + \|v\|_1 r^{1/2k}. \tag{4.14}$$

By Hölder's inequality

$$\int\limits_{R<r^{-1}}\frac{v(r',z')}{R^{1/2k}}d\nu' \le (\frac{8\pi^2 k}{4k-q})^{1/q}\|v\|_p r^{\frac{1}{2k}-\frac{1}{q}}. \tag{4.15}$$

Hence from (4.14) and (4.15) we find

$$G_1(r,z) \le (\frac{8\pi^2 k}{4k-q})^{1/q}\|v\|_p r^{\frac{1}{2k}-\frac{1}{q}} + \|v\|_1 r^{1/2k}. \tag{4.16}$$

The same calculation yields

$$G_2(r,z) \le (\frac{4\pi^2 k}{2k-q})^{1/q}\|v\|_p r^{\frac{1}{k}-\frac{1}{q}} + \|v\|_1 r^{1/k}. \tag{4.17}$$

Thus from (4.13), (4.16) and (4.17), we can find a positive number N depending on k and

q such that

$$Kv(r,z) \le N(\|v\|_1 + \|v\|_p)(r + r^{1+\frac{1}{2k}-\frac{1}{q}} + r^{1+\frac{1}{k}-\frac{1}{q}} + r^{1+\frac{1}{2k}} + r^{1+\frac{1}{k}}).$$

Since $q < 2k$, then it follows that $0 < 1 + \frac{1}{2k} - \frac{1}{q} < 1$. Therefore we have

$$Kv(r,z) \le M(\|v\|_1 + \|v\|_p)(r^{1+\frac{1}{2k}-\frac{1}{q}} + r^{1+\frac{1}{k}}),$$

where M is a positive number depending on N. $\square$

Remark 4.6. *By using [4, Lemma 2.8] we can show that for all $r > 0$, there is a constant C depending on p such that*

$$Kv(r,z) \le C(\|v\|_1 + \|v\|_p)r^2.$$

Lemma 4.7. *Let $k \ge 1$, let $\frac{2k}{2k-1} < p < \infty$, let q the exponent conjugate of p and let $0 < H < \infty$. Then there exists a positive number M' depending only on k, q and H such that*

$$Kv(r,z) \le M'(\|v\|_1 + \|v\|_p)(r^{1+\frac{1}{2k}-\frac{1}{q}} + r^{1+\frac{1}{k}})\min\{1, |z|^{\frac{-1}{2k+p}}\}$$

for all $(r,z) \in \Pi$, whenever $v \in L^1(\Pi,\nu) \cap L^p(\Pi,\nu)$ is Steiner-symmetric and $v(r,z) = 0$ for all $r > H$.

Proof. To prove this Lemma, we follow the same method as in [13, Lemma 5]. Indeed, if $w \in L^p(\Pi,\nu)$ is Steiner-symmetric and $|z| \ge b$ then

$$\int\limits_{|z-z'|<b} w(r',z')d\nu' \le \frac{b}{|z|}\int\limits_{\Pi} w(r',z')d\nu'. \tag{4.18}$$

We assume that $v \in L^1(\Pi,\nu) \cap L^p(\Pi,\nu)$ is Steiner-symmetric, let $(r,z) \in \Pi$ be a fixed point and for $(r',z') \in \Pi$, we define the function v_1 as follows

$$v_1(r',z') = \begin{cases} v(r',z') & \text{if } |z-z'| < |z|^{\frac{2k}{2k+p}} \\ 0 & \text{if } |z-z'| \ge |z|^{\frac{2k}{2k+p}}. \end{cases}$$

By using Lemma 4.5 and (4.18)

$$\begin{aligned} Kv_1(r',z') &\le M(r^{1+\frac{1}{k}} + r^{1+\frac{1}{2k}-\frac{1}{q}})(\|v_1\|_1 + \|v_1\|_p) \\ &\le M(r^{1+\frac{1}{k}} + r^{1+\frac{1}{2k}-\frac{1}{q}})(\frac{|z|^{\frac{2k}{2k+p}}}{|z|}\|v\|_1 + (\frac{|z|^{\frac{2k}{2k+p}}}{|z|})^{1/p}\|v\|_p) \\ &\le M(\|v\|_1 + \|v\|_p)(r^{1+\frac{1}{k}} + r^{1+\frac{1}{2k}-\frac{1}{q}})(|z|^{-\frac{p}{2k+p}} + |z|^{\frac{-1}{2k+p}}). \end{aligned} \tag{4.19}$$

We recall that $R = ((r-r')^2 + (z-z')^2)^{1/2}$, $\overline{R} = ((r+r')^2 + (z-z')^2)^{1/2}$ and $\overline{R} > r'$. By

Lemma 4.3 we have, for $v_2(r,z) = v(r,z) - v_1(r,z)$

$$Kv_2(r,z) \leq \frac{\log 9}{2\pi^2} \int\limits_{\Pi} \frac{rr'}{\overline{R}} v_2(r',z')d\nu' + \frac{1}{2\pi^2} \int\limits_{\Pi} r \log \frac{\overline{R}}{R} v_2(r',z')d\nu'. \tag{4.20}$$

We set

$$F_1(r,z) = \int\limits_{\Pi} \frac{rr'}{\overline{R}} v_2(r',z')d\nu' \quad \text{and} \quad F_2(r,z) = \int\limits_{\Pi} r \log \frac{\overline{R}}{R} v_2(r',z')d\nu'.$$

For $F_1(r,z)$ we have

$$F_1(r,z) = \int\limits_{|z-z'|\geq |z|^{\frac{2k}{2k+p}}} \frac{rr'}{\overline{R}} v(r,z)d\nu' \leq H \int\limits_{|z-z'|\geq |z|^{\frac{2k}{2k+p}}} r\frac{v(r',z')}{|z-z'|}d\nu' \leq H\|v\|_1 r|z|^{\frac{-2k}{2k+p}}. \tag{4.21}$$

Now for all r and r' we have $(rr')^{1/2k} \leq (rR + r^2)^{1/2k} \leq r^{1/2k}R^{1/2k} + r^{1/k}$, thus by using (4.12) we find

$$\begin{aligned}
F_2(r,z) &\leq 2^{1/k}kr \int\limits_{|z-z'|>|z|^{\frac{2k}{2k+p}}} \frac{(rr')^{1/2k}}{R^{1/k}} v_2(r,z)d\nu' \\
&\leq 2^{1/k}k(r^{1+\frac{1}{2k}} \int\limits_{|z-z'|>|z|^{\frac{2k}{2k+p}}} \frac{v(r',z')}{R^{1/2k}}d\nu' + r^{1+\frac{1}{k}} \int\limits_{|z-z'|>|z|^{\frac{2k}{2k+p}}} \frac{v(r',z')}{R^{1/k}}d\nu') \\
&\leq 2^{1/k}k(r^{1+\frac{1}{2k}}|z|^{\frac{-1}{2k+p}} + r^{1+\frac{1}{k}}|z|^{\frac{-2}{2k+p}})\|v\|_1 \\
&\leq 2^{1/k}k(r^{1+\frac{1}{2k}} + r^{1+\frac{1}{2k}-\frac{1}{q}} + r^{1+\frac{1}{k}} + r^{1+\frac{1}{k}-\frac{1}{q}})(|z|^{\frac{-1}{2k+p}} + |z|^{\frac{-2}{2k+p}})\|v\|_1 \\
&\leq 2^{1/k}k(r^{1+\frac{1}{k}} + r^{1+\frac{1}{2k}-\frac{1}{q}})(|z|^{\frac{-1}{2k+p}} + |z|^{\frac{-2}{2k+p}})\|v\|_1.
\end{aligned} \tag{4.22}$$

Hence it follows from (4.20), (4.21) and (4.22) that there exists a positive constant $M_1 > 0$ depending on k and H such that

$$Kv_2(r,z) \leq M_1\|v\|_1(r + r^{1+\frac{1}{2k}-\frac{1}{q}} + r^{1+\frac{1}{k}})(|z|^{\frac{-2k}{2k+p}} + |z|^{\frac{-1}{2k+p}} + |z|^{\frac{-2}{2k+p}}). \tag{4.23}$$

Therefore from (4.20) and (4.23) we can choose M' depending only on N and M_1 such that

$$Kv(r,z) \leq M'(\|v\|_1 + \|v\|_p)(r^{1+\frac{1}{k}} + r^{1+\frac{1}{2k}-\frac{1}{q}}) \min\{1, |z|^{\frac{-1}{2k+p}}\}.$$

This completes the proof. $\square$

The last Lemma in this section concerns the operator K. For a bounded domain Ω in Π, we recall that the Hilbert space $\mathcal{H}$ was defined to be the completion of $\mathcal{D}(\Omega)$ with scalar product

$$\langle u,v\rangle = \int\limits_{\Omega} \frac{1}{r^2}\nabla u.\nabla v d\nu.$$

Lemma 4.8. *Let $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$, and let $\Omega \subset \Pi$ be bounded domain. Then for $v \in L^p(\Omega, \nu)$ there is a unique $Kv \in \mathcal{H}$ that is a weak solution of $\mathcal{L}u = v$ in Ω. Furthermore*

1. *$K : L^p(\Omega, \nu) \to \mathcal{H}$ is a bounded linear operator,*
2. *$K : L^p(\Omega, \nu) \to L^q(\Omega, \nu)$ is symmetric, strictly positive and compact operator,*
3. *If $v \in L^p(\Omega, \nu)$ then $Kv \in W^2_{loc}(\Omega)$.*

See [10, Lemma 8].

4.4 Properties of the functional Φ^n_λ

As we did in Chapter 2, in this section all the estimates of the function Kv that were proved in the last section, will be used to derive some properties for the functional Φ^n_λ. Throughout this section we shall use 1_Ω to denote the characteristic function of a subset Ω in Π defined by $1_\Omega(r, z) = 1$ if $(r, z) \in \Omega$ and $1_\Omega(r, z) = 0$ if $(r, z) \notin \Omega$.

Lemma 4.9. *Let $n \geq 1$, let $\lambda > 0$, let $k \geq 1$, let $\frac{2k}{2k-1} < p < \infty$ and let $v_0 \in L^p(\Pi, \nu)$ be a non-negative function having support of finite volume. Let $\mathcal{R}(v_0)$ be the set of rearrangements of v_0 having bounded support in Π with respect to ν measure. Then there exists a positive number X that may depend on λ and $\|v_0\|_p$, such that if $v \in \mathcal{R}(v_0)$ and does not vanish almost everywhere for $r > X$, we can choose a positive number ξ for which we have*

(i) $\Phi^n_\lambda(v_1) > \Phi^n_\lambda(v)$

(ii) $\Phi^n_\lambda(v_2) > \Phi^n_\lambda(v)$,

where $v_1 = v 1_{\{r<X\}}$ and $v_2 \in \mathcal{R}(v_0)$ is some function such that $\operatorname{supp} v_2 \subset \Pi(\xi, X)$.

Proof. For (i), from Lemma 4.5 we can choose a positive constant M depending only on $\nu(\operatorname{supp} v_0)$, p and k such that

$$Kv(r, z) \leq M\|v_0\|_p r^{1+\frac{1}{k}}$$

for all $r > 1$. Then it follows that

$$Kv(r, z) - \frac{\lambda}{2n} r^{2n} \leq M\|v_0\|_p r^{1+\frac{1}{k}} - \frac{\lambda}{2n} r^{2n},$$

hence we can take $X(\lambda) = (\frac{2nM\|v_0\|_p}{\lambda})^{\frac{k}{(2n-1)k-1}}$ such that if $r > X = \max\{1, X(\lambda)\}$, then

$$Kv(r, z) - \frac{\lambda}{2n} r^{2n} < 0. \tag{4.24}$$

Now for all $v \in L^p(\Pi, \nu)$, we write

$$v(r, z) = v_1(r, z) + \widetilde{v}(r, z),$$

where $\widetilde{v} = v 1_{\mathbb{R}\times(X,\infty)}$. From Lemma 4.8, K is a strictly positive operator. Hence Φ^n_λ is strictly convex; then by assuming that $\widetilde{v} \neq 0$, applying (4.24) and convexity of Φ^n_λ we find

$$\begin{aligned}\Phi^n_\lambda(v_1) &= \Phi^n_\lambda(v-\widetilde{v}) \\ &> \Phi^n_\lambda(v) + \int_\Pi (-Kv(r,z) + \frac{\lambda}{2n} r^{2n})\widetilde{v}(r,z)d\nu > \Phi^n_\lambda(v).\end{aligned}$$

To prove (ii), we assume that v does not vanish almost everywhere for $r > X$. Let $a > 0$ be such that $\nu(\text{supp } v_0) = 2\pi^2 a^3$. Let ξ and ε be two positive numbers chosen such that $\xi\varepsilon^2 \geq \pi a^3$ and $\varepsilon < X$; we can then find a measurable subset A that satisfies $A \subset (-\xi,\xi)\times(0,\varepsilon)$, $A \cap \text{supp } v_1 = \emptyset$ and finally $\nu(A) = \nu(\text{supp } \widetilde{v})$. Now the Theorem of isomorphism of measure spaces [27] shows that there exists an isomorphism $\sigma : A \to \text{supp } \widetilde{v}$, hence by setting $\bar{v} = \widetilde{v}\circ\sigma$ we find that $\bar{v}$ is rearrangement of $\widetilde{v}$ supported by the measurable set A. Thus we have

$$\text{supp } \bar{v} \cap \text{supp } v_1 = \emptyset$$

and moreover, the function v_2 that is defined as $v_2 = v_1 + \bar{v}$ is a rearrangement of v. It remains just to show that, if ε is small enough, then

$$\Phi^n_\lambda(v_2) > \Phi^n_\lambda(v).$$

To do that, we need just to show that

$$\Phi^n_\lambda(v_2) - \Phi^n_\lambda(v_1) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Indeed, from the form of v_2 we have

$$\begin{aligned}\Phi^n_\lambda(v_2) &= \Phi^n_\lambda(v_1+\bar{v}) \\ &= \Phi^n_\lambda(v_1) + \Phi^n_\lambda(\bar{v}) + \frac{1}{2}\int_\Pi v_1(r,z)K\bar{v}(r,z)d\nu + \frac{1}{2}\int_\Pi \bar{v}(r,z)Kv_1(r,z)d\nu. \quad (4.25)\end{aligned}$$

By Lemma 4.8, K is symmetric; then from (4.25) we find

$$\Phi^n_\lambda(v_2) - \Phi^n_\lambda(v_1) < \int_\Pi v_1(r,z)K\bar{v}(r,z)d\nu + \frac{1}{2}\int_\Pi \bar{v}(r,z)K\bar{v}(r,z)d\nu.$$

Also, by Lemma 4.8, K is a positive operator; hence it follows that

$$\Phi^n_\lambda(v_2) - \Phi^n_\lambda(v_1) \geq -\frac{\lambda}{2n}\int_\Pi r^{2n}\bar{v}(r,z)d\nu. \tag{4.26}$$

By using Remark 4.6 and (4.26), we get

$$-\frac{\lambda}{2n}\|\bar{v}\|_1\varepsilon^{2n} \leq \Phi^n_\lambda(v_2) - \Phi^n_\lambda(v_1) < C_{a,p}(\|\bar{v}\|_p\|\|v_1\|_1 + \frac{1}{2}\|\bar{v}\|_p\|\bar{v}\|_1)\varepsilon^2,$$

where $C_{a,p}$ is a positive number depending on a and p. Thus

$$\Phi_\lambda^n(v_2) - \Phi_\lambda^n(v_1) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Therefore we deduce that for small $\varepsilon > 0$

$$\Phi_\lambda^n(v_2) > \Phi_\lambda^n(v),$$

which completes the proof. $\square$

Lemma 4.10. *Let the assumptions about λ, n, k, p, v_0 and $\mathcal{R}(v_0)$ be the same as in Lemma 4.9. Let $\mathcal{F}(v_0)$ be the set of rearrangements of v_0 on Π. Then we have*

$$\sup_{\mathcal{F}(v_0)} \Phi_\lambda^n = \sup_{\mathcal{R}(v_0)} \Phi_\lambda^n.$$

Proof. To prove this Lemma, we follow the same argument that we have used in proving Lemma 4.9. Thus, we should prove firstly that for a given rearrangement v in $\mathcal{F}(v_0)$, there exists a function $\bar{v} \in \mathcal{F}(v_0)$ supported by the region $\mathbb{R} \times (0, X)$ such that

$$\Phi_\lambda^n(\bar{v}) \geq \Phi_\lambda^n(v),$$

where X is a positive number that is defined in Lemma 4.9. The second step is to show that for arbitrary $\delta > 0$, there exists a rearrangement v^* of $v - \widetilde{v}$ supported in the region $\Pi(Z, X) \backslash \text{supp } v$, where $\widetilde{v} = v 1_{\Pi(Z,X)}$ for some $Z > 0$ such that

$$\Phi_\lambda^n(v^* + \widetilde{v}) \geq \Phi_\lambda^n(v) - \delta.$$

We set $\bar{v} = v 1_{\mathbb{R}\times(0,X)}$, then by calculating as in Lemma 4.9 (i), we get

$$\Phi_\lambda^n(\bar{v}) \geq \Phi_\lambda^n(v).$$

For the second step, we firstly build v^*. Indeed, let $0 < \varepsilon < X$ and Z be two positive numbers chosen so that $Z\varepsilon^2 > \pi a^3$, where $\nu(\text{supp } v_0) = 2\pi^2 a^3$; so there exists a measurable set B that satisfies $B \subset \Pi(Z, \varepsilon)$, $B \cap \text{supp } v 1_{\Pi(Z,\varepsilon))} = \emptyset$ and $\nu(B) = 2\pi^2 a^3 - \nu(\text{supp } v \cap \Pi(Z, \varepsilon))$. Thus it follows from the theorem of isomorphism of measure spaces [27] that there is an isomorphism $\sigma : \text{supp } v 1_{\Pi\backslash\Pi(Z,\varepsilon)} \to B$; hence by setting $v^* = v \circ \sigma^{-1}$, we find that v^* is a rearrangement of $v - \widetilde{v}$ supported by the measurable set B, where $\widetilde{v} = v 1_{\Pi(Z,X)}$. It remains just to prove that, for $\delta > 0$ arbitrary, by suitable choice of ε and Z we can ensure that

$$\Phi_\lambda^n(v^* + \widetilde{v}) \geq \Phi_\lambda^n(v) - \delta.$$

Indeed, we can write $\bar{v}$ as $\bar{v} = \widetilde{v} + v_*$, where $v_* = v 1_{\mathbb{R}\times(0,X)\backslash\Pi(Z,X)}$. Then using the fact that K is symmetric, we get

$$\begin{aligned}\Phi_\lambda^n(\bar{v}) &= \Phi_\lambda^n(\widetilde{v}+v_*)\\ &= \Phi_\lambda^n(\widetilde{v})+\Phi_\lambda^n(v_*)+\int\limits_\Pi v_*K\widetilde{v}d\nu.\end{aligned} \tag{4.27}$$

Also

$$\Phi_\lambda^n(\widetilde{v}+v^*)=\Phi_\lambda^n(\widetilde{v})+\Phi_\lambda(v^*)+\int\limits_\Pi v^*K\widetilde{v}d\nu.$$

If now we set

$$D\Phi_\lambda^n(\bar{v},\widetilde{v},v^*)=\Phi_\lambda^n(\bar{v})-\Phi_\lambda^n(\widetilde{v}+v^*),$$

then by using Remark 4.6 and (4.27) we get

$$\begin{aligned}D\Phi_\lambda^n(\bar{v},\widetilde{v},v^*) &= \Phi_\lambda^n(v_*)-\Phi_\lambda^n(v^*)+\int\limits_\Pi(v_*-v^*)K\widetilde{v}d\nu\\ &\le \frac{1}{2}\int\limits_\Pi v_*Kv_*d\nu+\frac{\lambda}{2n}\int\limits_\Pi r^{2n}v^*+\int\limits_\Pi v_*K\widetilde{v}d\nu\\ &\le \frac{C_{a,p}\|v_*\|_p}{2}\int\limits_\Pi r^2v_*d\nu+\frac{\lambda\|v^*\|_1}{2n}\varepsilon^{2n}+C_{a,p}\|v_*\|_p\int\limits_\Pi r^2\widetilde{v}d\nu\\ &\le \frac{C_{a,p}\|v_*\|_p}{2}\int\limits_\Pi r^2\bar{v}d\nu+\frac{\lambda\|v_0\|_1}{2n}\varepsilon^{2n}+C_{a,p}\|v_*\|_p\|v_0\|_1X^2\\ &\le \frac{3C_{a,p}X^2\|v_0\|_1\|v_*\|_p}{2}+\frac{\lambda\|v_0\|_1}{2n}\varepsilon^{2n},\end{aligned} \tag{4.28}$$

where $C_{a,p}$ is a positive constant depending on p and a. Note that $\|v_*\|_1\to 0$ and $\|v_*\|_p\to 0$ as $Z\to\infty$. Then for given $\varepsilon>0$ we can find $Z_0=Z_0(\varepsilon)>0$ with $Z_0\varepsilon^2\ge\pi a^3$ such that if $Z\ge Z_0$, then we have

$$\|v_*\|_p\le\frac{\varepsilon^{2n}}{X^2}.$$

Thus for all $Z\ge Z_0$ we have

$$D\Phi_\lambda^n(\bar{v},\widetilde{v},v^*)\le(\frac{3C_{a,p}\|v_0\|_1}{2}+\frac{\lambda}{2n})\varepsilon^{2n}.$$

For $\delta>0$ arbitrary, we set

$$\varepsilon_0=(\frac{2n\delta}{3nC_{a,p}\|v_0\|_p+\lambda})^{1/2n};$$

then it follows from (4.28)

$$\Phi_\lambda^n(\bar{v})-\Phi_\lambda^n(v^*+\widetilde{v})\le\delta$$

for all $\varepsilon\le\varepsilon_0$ and $Z\ge Z_0$. Therefore by using the first step we deduce that for all $\delta>0$

arbitrary, we can choose a rearrangement $v^* + \widetilde{v}$ of v having bounded support such that

$$\Phi_\lambda^n(v^* + \widetilde{v}) \geq \Phi_\lambda^n(v) - \delta.$$

Hence the Lemma is proved. $\square$

Lemma 4.11. *Let $n > 2$, let $\lambda > 0$, let $1 < p < \infty$ and let $v_0 \in L^p(\Pi, \nu)$ be a non-negative function having support of finite volume. Let t be a positive number chosen small enough that $\nu(v_0^{-1}(t, \infty)) > 0$. Let N be a positive number such that $\pi N^3 = \nu(v_0^{-1}(t, \infty))$. Then a rearrangement v of v_0 can be chosen such that $v(r,z) \geq t1_{[0,N]^2}(r,z)$. Furthermore, if we define v_ρ by*

$$v_\rho(r,z) = v(\rho^{-1}r, \rho^2 z)$$

for all $(r,z) \in \Pi$, where $\rho \in (0,1]$, then we have

$$\Phi_\lambda^n(v_\rho) > 0$$

for all sufficiently small positive ρ.

Proof. Let l be a positive number such that $\nu(\text{supp } v_0) = \pi l^3$. For small $t > 0$ we set

$$S_t = \{(r,z) \in \Pi | v_0(r,z) > t\};$$

then $S_t \subset \text{supp } v_0$. By using the Theorem of isomorphism of measure spaces [27], there is an isomorphism $\sigma : [0,N]^3 \to S_t$. Let τ denote a measure-preserving bijection defined from $[0,l]^2 \backslash [0,N]^2$ into $\text{supp } v_0 \backslash S_t$, and let v be the function defined by

$$v(r,z) = \begin{cases} v_0 \circ \sigma(r,z) & \text{if } (r,z) \in [0,N]^2 \\ v_0 \circ \tau(r,z) & \text{if } (r,z) \in [0,l]^2 \backslash [0,N]^2. \end{cases}$$

Then v is a non-negative function and also is a rearrangement of v_0. Moreover

$$v(r,z) \geq t1_{[0,N]^2}(r,z).$$

It remains now just to show that for small $\rho > 0$

$$\Phi_\lambda^n(v_\rho) > 0.$$

Indeed, for v_ρ, we have

$$\Phi_\lambda^n(v_\rho) = E(v_\rho) - \frac{\lambda}{2n} I_n(v_\rho).$$

Then by making the linear change of variable $r' = \rho^{-1}r$ and $z' = \rho^2 z$ which preserves ν, we get

$$I_n(v_\rho) = I_{2n}(v)\rho^{2n}. \tag{4.29}$$

By using Lemma 4.3, we have

$$
\begin{aligned}
E(v_\rho) &= \frac{1}{2}\int_\Pi\int_\Pi G(r,r',z,z')v_\rho(r,z)v_\rho(r',z')d\nu' d\nu \\
&\geq \frac{1}{12\pi^2}\int_\Pi\int_\Pi \frac{(rr')^2}{((r+r')^2+(z-z')^2)^{3/2}} v(\rho^{-1}r,\rho^2 z)v(\rho^{-1}r',\rho^2 z')d\nu' d\nu \\
&= \frac{1}{12\pi^2}\int_\Pi\int_\Pi \frac{\rho^4(rr')^2}{(\rho^2(r+r')^2+\rho^{-4}(z-z')^2)^{3/2}} v(r,z)v(r',z')d\nu' d\nu \\
&= \frac{1}{12\pi^2}\int_\Pi\int_\Pi \frac{\rho^{10}(rr')^2}{(\rho^6(r+r')^2+(z-z')^2)^{3/2}} v(r,z)v(r',z')d\nu' d\nu \\
&\geq \frac{1}{12\pi^2}\int\int_{\Pi\times\Pi,|z-z'|<\rho^3} \frac{\rho^{10}(rr')^2}{(\rho^6(r+r')^2+(z-z')^2)^{3/2}} v(r,z)v(r',z')d\nu' d\nu \\
&\geq \frac{t^2\rho}{3}\int\int_{\Pi\times\Pi,|z-z'|<\rho^3} \frac{(rr')^3}{((r+r')^2+1)^{3/2}} 1_{[0,N]^2}(r,z)1_{[0,N]^2}(r',z')dr'drdz'dz \\
&= \frac{t^2\rho}{3}\int_0^N\int_0^N \frac{(rr')^3 dr'dr}{(1+(r+r')^2)^{3/2}}\int\int_{0\leq z\leq N, 0\leq z'\leq N,|z-z'|<\rho^3} dz'dz. \\
&\geq \frac{N^8t^2\rho}{48(4N^2+1)^{3/2}}\int\int_{0\leq z'\leq N, 0\leq z\leq N,|z-z'|<\rho^3} dz'dz. \qquad (4.30)
\end{aligned}
$$

We now set

$$C_t(N) = \frac{N^8t^2}{48(4N^2+1)^{3/2}}$$

and

$$J_N(\rho) = \int_{A(\rho)} dz'dz,$$

where

$$A(\rho) = \{(r,z)|0\leq z\leq N, 0\leq z'\leq N, |z-z'|<\rho^3\}.$$

By following the same calculations as in Lemma 2.14, then from (4.30) we find

$$J_N(\rho) = 2N\rho^3 - \rho^6 \qquad (4.31)$$

provided $\rho^3 < N$. Thus it follows from (4.29) and (4.31) that

$$\Phi_\lambda^n(v_\rho) \geq 2NC_t(N)\rho^4 - C_t(N)\rho^7 - \lambda M(v)\rho^{2n} > 0$$

for small positive ρ, because $n > 2$. □

The next Lemmas in this section are analogous to [17, Lemma 3 and 4].

Lemma 4.12. *Let $n\geq 1$, let $\lambda>0$, let $2<p<\infty$ and let $v_0\in L^1(\Pi,\nu)\cap L^p(\Pi,\nu)$ be a non-negative. Let X be positive number defined as in Lemma 4.9. Let α and ξ be* *two positive numbers. Then there exists a positive number β, which may depend on ξ and α, such that for every $v \in L^1(\Pi,\nu) \cap L^p(\Pi,\nu)$ that satisfies $\Phi^n_X(v) \geq \alpha$, $\|v\|_p = \|v_0\|_p$, $\|v\|_1 = \|v_0\|_1$ and* supp $v \subset \mathbb{R} \times (0,X)$*, there is a square $S(\xi) \subset \mathbb{R} \times (0,X)$ of side ξ for which we have*

$$\|v 1_{S(\xi)}\|_p > \beta.$$

Proof. Let $1 < q < 2$ be the conjugate exponent of p. Let $v \in L^1(\Pi,\nu) \cap L^p(\Pi,\nu)$ be such that $\Phi^n_X(v) \geq \alpha$, $\|v\|_p = \|v_0\|_p$, $\|v\|_1 = \|v_0\|_1$ and supp $v \subset \mathbb{R} \times (0,X)$. For a fixed $\beta > 0$ and for every square $S(\xi)$ of side ξ we assume that

$$\|v 1_{S(\xi)}\|_p \leq \beta. \tag{4.32}$$

For a fixed point $(r,z) \in \mathbb{R}\times(0,X)$, we consider the square $S(N\xi)$ of centre $(r,z) \in \mathbb{R}\times(0,X)$ and side $N\xi$, where N is positive integer. Then we can cover $S(N\xi)\cap\mathbb{R}\times(0,X)$ by a number N^2 of disjoint squares $\{C_j(\xi)\}_{j=1}^{N^2} \subset \mathbb{R} \times (0,X)$ of side ξ. We set $\overline{R}^2 = (r+r')^2 + (z-z')^2$ and $R^2 = (r-r')^2 + (z-z')^2$. Then using the fact that supp $v \subset \mathbb{R} \times (0,X)$ and Lemma 4.3 we have

$$G(r,r',z,z') \leq \frac{rr'}{2\pi^2\overline{R}} \log \frac{9\overline{R}}{R} \leq \frac{9rr'}{2\pi^2 R},$$

hence we find

$$\begin{aligned}
Kv(r,z) &= \left(\int\limits_{S(N\xi)\cap\mathbb{R}\times(0,X)} + \int\limits_{\mathbb{R}\times(0,X)\setminus S(N\xi)} \right) G(r,r',z,z')v(r',z')d\nu' \\
&\leq \frac{9}{2\pi^2} \left(\sum_{j=1}^{j=N^2} \int\limits_{C_j(\xi)} + \int\limits_{R>\frac{1}{2}N\xi} \right) \frac{rr'}{R} v(r',z')d\nu' \\
&\leq \frac{9}{2\pi^2} \left(Xr \sum_{j=1}^{j=N^2} \int\limits_{C_j(\xi)} \frac{v(r',z')}{R} d\nu' + \frac{2Xr\|v_0\|_1}{N\xi} \right).
\end{aligned} \tag{4.33}$$

We set

$$G_j(r,z) = \int\limits_{C_j(\xi)} \frac{v(r',z')}{R} d\nu'.$$

Then it follows from Hölder's inequality that

$$G_j(r,z) \leq \|v 1_{C_j(\xi)}\|_p \left(\int\limits_{C_j(\xi)} \frac{d\nu'}{R^q} \right)^{1/q}. \tag{4.34}$$

Now let $D(N\xi)$ be a disc of centre (r,z) and radius $\frac{\sqrt{2}}{2}N\xi$, then we get

$$\int\limits_{C_j(\xi)} \frac{d\nu'}{R^q} \le \int\limits_{D(\xi)} \frac{d\nu'}{R^q} = \frac{4\pi^2}{2-q}(\frac{\sqrt{2}}{2}\xi)^{2-q}N^{2-q}r. \tag{4.35}$$

Hence it follows from (4.32), (4.34) and (4.35) that

$$G_j(r,z) \le (\frac{4\pi^2}{2-q})^{1/q}(\frac{\sqrt{2}}{2}\xi)^{\frac{2-q}{q}}N^{\frac{2-q}{q}}\beta r^{1/q}. \tag{4.36}$$

Now using (4.33) and (4.36) we have

$$\begin{aligned} Kv(r,z) &\le \frac{9Xr}{2\pi^2}(N^2(\frac{4\pi^2}{2-q})^{1/q}(\frac{\sqrt{2}}{2}\xi)^{\frac{2-q}{q}}\beta N^{\frac{2-q}{q}}r^{1/q} + \frac{2\|v_0\|_1}{N\xi}) \\ &= \frac{9Xr}{2\pi^2}((\frac{4\pi^2}{2-q})^{1/q}(\frac{\sqrt{2}}{2}\xi)^{\frac{2-q}{q}}N^{\frac{2+q}{q}}r^{1/q} + \frac{2\|v_0\|_1}{N\xi}) \\ &\le \frac{9Xr}{2\pi^2}((\frac{4\pi^2}{2-q})^{1/q}(\frac{\sqrt{2}}{2}\xi)^{\frac{2-q}{q}}N^4 r^{1/q} + \frac{2\|v_0\|_1}{N\xi}), \end{aligned} \tag{4.37}$$

because $\frac{q+2}{q} < 4$. Therefore from (4.37) we have

$$\frac{1}{2}\int\limits_{r<X} v(r,z)Kv(r,z)d\nu \le M\beta N^4 + \frac{9\|v_0\|_1^2X^2}{\pi^2 N\xi},$$

where

$$M = \frac{9X^{\frac{2q+1}{q}}}{2\pi^2}(\frac{4\pi^2}{2-q})^{1/q}(\frac{\sqrt{2}}{2}\xi)^{\frac{2-q}{q}}\|v_0\|_1.$$

By choosing N large enough such that $\frac{9\|v_0\|_1^2X^2}{\pi^2N\xi} < \alpha/2$ (it does not matter how small λ is), and also by choosing β small enough so that $M\beta N^4 < \alpha/2$; we get $\Phi_\lambda^n(v) < \alpha$ and this is a contradiction. Hence there exists a constant β depending on α, X and ξ, and there exists a square $S(\xi)$ of side ξ for which we have

$$\|v1_{S(\xi)}\|_p > \beta.$$

This completes the proof. □

Lemma 4.13. *Let the assumptions about n, λ, X, p, α, ξ and v_0 be the same as in Lemma 4.12. Then there exist a positive number ε such that for every $v \in L^1(\Pi,\nu) \cap L^p(\Pi,\nu)$ that satisfies $\Phi_\lambda^n(v) \ge \alpha$, $\|v\|_1 = \|v_0\|_1$, $\|v\|_p = \|v_0\|_p$ and $\operatorname{supp} v \subset \mathbb{R}\times(0,X)$, there is a square of $S(\xi)$ of side ξ for which*

$$\nu(\{(r,z) \in S(\xi) | v(r,z) > 0\}) \ge \varepsilon.$$

Furthermore, there exist a positive number η depending on β and ε such that

$$\int\limits_{S(\xi)} r^2 v(r,z)d\nu > \eta.$$

Proof. Let $\beta > 0$ be the number provided by Lemma 4.12. Let $v \in L^1(\Pi,\nu) \cap L^p(\Pi,\nu)$ satisfy $\Phi^{\eta}_{\lambda}(v) \geq \alpha$, $\|v\|_1 = \|v_0\|_1$, $\|v\|_p = \|v_0\|_1$ and supp $v \subset \mathbb{R}\times(0,X)$. Then for the given ξ there is a square $S(\xi)$ of side ξ, such that

$$\int\limits_{S(\xi)} |v(r,z)|^p d\nu > \beta^p.$$

By setting

$$V = \{(r,z) \in S(\xi) | v(r,z) > 0\};$$

then we have

$$\int\limits_{\Pi} |v(r,z)|^p 1_V(r,z)d\nu = \int\limits_{S(\xi)} |v(r,z)|^p d\nu > \beta^p.$$

If we denote by $v^{\triangle}$ and $1^{\triangle}_V$ the decreasing rearrangements of v and 1_V respectively, then by using the basic rearrangement inequality we have

$$\int\limits_{\Pi} |v(r,z)|^p 1_V(r,z)d\nu \leq \int\limits_0^{\infty} |v^{\triangle}(s)|^p 1^{\triangle}_V(s)dt = \int\limits_0^{\nu(V)} |v^{\triangle}(s)|^p ds = \int\limits_0^{\nu(V)} |v_0^{\triangle}(s)|^p ds,$$

where $v_0^{\triangle}$ is the decreasing rearrangement of v_0. Hence we find

$$\int\limits_0^{\nu(V)} |v_0^{\triangle}(s)|^p ds > \beta^p.$$

By using Lebesgue's dominated convergence theorem [27, theorem D], we have

$$\int\limits_0^{\Lambda} |v_0^{\triangle}(s)|^p ds \to 0 \quad \text{as} \quad \Lambda \to 0,$$

then there exists a positive number ε such that

$$\int\limits_0^{\varepsilon} |v_0^{\triangle}(s)|^p ds < \beta^p.$$

Therefore

$$\nu(\{(r,z) \in S(\zeta) | v(r,z) > 0\}) \geq \varepsilon.$$

Let δ be a positive number chosen such that $\pi\delta^2\xi < \varepsilon/2$, and let $\mathcal{R}$ be the rectangle that is defined by

$$\mathcal{R} = [r' - \xi/2 + \delta, r' + \xi/2] \times [z' - \xi/2, z' + \xi/2],$$

where (r', z') is the centre of $S(\xi)$. Then for all $(r, z) \in \Pi$ we have

$$r1_{S(\xi)}(r, z) \geq r1_D(r, z) \geq \delta 1_D(r, z),$$

where $D = \mathcal{R} \cap \operatorname{supp} v$. Thus

$$\int\limits_{S(\xi)} r^2 v(r, z) d\nu \geq \int\limits_{S(\xi)} \delta^2 1_D(r, z) v(r, z) d\nu = \delta^2 \int\limits_{D\cap S(\xi)} v(r, z) d\nu.$$

Let v^∇ and v_0^∇ be the increasing rearrangements of v and v_0 respectively on $[0, \nu(\operatorname{supp} v_0)]$. Since $D \cap S(\xi) = D$, then we get

$$\int\limits_{D\cap S(\xi)} v(r, z) d\nu = \int\limits_{D} v(r, z) d\nu \geq \int\limits_0^{\nu(D)} v^\nabla(s) ds = \int\limits_0^{\nu(D)} v_0^\nabla(s) ds.$$

The fact that $\nu(D) \geq \varepsilon/2$ ensures that

$$\int\limits_{S(\xi)} r^2 v(r, z) d\nu > \eta,$$

where $\eta = \delta^2 \int\limits_0^{\varepsilon/2} v_0^\nabla(s) ds.$ □

Lemmas 4.12 and 4.13 show that the support of any maximiser for the functional Φ_λ^n has a significant intersection with an unknown square of given side, assuming only $n \geq 1$. In order to prove the main Theorem we will use these Lemmas only in the case when $n \geq 4$, because in this case, the measure of the set where $Kv - \frac{\lambda}{2n} r^{2n} > 0$ has infinite volume, and therefore the dependence η on λ will not have any effect. However, the next Lemma will show us that even if $2 \leq n < 4$, then under an assumption that we believe is true (we will discuss this matter later), we can show that the volume of the set where $Kv - \frac{\lambda}{2n} r^{2n} > 0$ tends to ∞ as $\lambda \to 0$, where v is any maximiser of Φ_λ^n over the set $\mathcal{F}(\xi, X)$ defined as follows

$$\mathcal{F}(\xi, X) = \{v \in \mathcal{F}(v_0) | \operatorname{supp} v \subset \Pi(\xi, X)\},$$

where X is a positive number defined as in Lemma 4.9, and ξ is large positive number.

Lemma 4.14. *Let $\alpha > 0$, let $X > 1$, let $\frac{2k}{2k-1} < p < \infty$ and let $v \in L^1(\Pi, \nu) \cap L^p(\Pi, \nu)$ be a non-negative function. Let $\{X\}_{i=0}^{i=i_*}$ be a positive sequence in $\mathbb{R}$ such that $X_i \leq X_j$ for all $i \leq j$ and $X_{i_*} = X$. Then there exists a positive number m depending on $\|v\|_1$, $\|v\|_p$, k and* *p such that if v satisfies*

$$\int_{|z|<X_0, r<X} v(r,z)Kv(r,z)d\nu \geq \alpha,$$

then

$$\int_{|z|<X_0, r<X_0} r^2 v(r,z)d\nu + \sum_{i=0}^{i=i_*} \int_{|z|<X_0, X_{i-1}\leq r<X_i} v(r,z)d\nu \geq \frac{\alpha}{m} X^{-\frac{k+1}{k}}.$$

Proof. We may write

$$\int_{|z|<X_0, r<X} vKvd\nu = \left(\int_{|z|<X_0, r<X_0} + \sum_{i=1}^{i=n_*} \int_{|z|<X_0, X_{i-1}\leq r<X_i} \right) vKvd\nu. \tag{4.38}$$

By using Remark 4.6, we can choose a positive constant C depending on p such that

$$Kv(r,z) \leq C(\|v\|_1 + \|v\|_p) r^2;$$

hence it follows that

$$\int_{|z|<X_0, r<X_0} vKvd\nu \leq C(\|v\|_1 + \|v\|_p) \int_{|z|<X_0, r<X_0} r^2 vd\nu. \tag{4.39}$$

Applying now Lemma 4.5 we get

$$\int_{|z|<X_0, X_{i-1}\leq r<X_i} vKvd\nu \leq N(\|v\|_1 + \|v\|_p) X^{1+\frac{1}{k}} \int_{|z|<X_0, r<X_{i-1}\leq r<X_i} vd\nu, \tag{4.40}$$

where N is a positive constant depending on p and k. Since $X > 1$ then by (4.39) and (4.40) we find that

$$\int_{|z|<X_0, r<X} vKvd\nu \leq mX^{\frac{k+1}{k}} \left(\int_{|z|<X_0, r<X_0} r^2 vd\nu + \sum_{i=1}^{i=n_*} \int_{|z|<X_0, X_{i-1}\leq r<X_i} vd\nu \right), \tag{4.41}$$

where

$$m = \max\{C, N\}(\|v\|_1 + \|v\|_p).$$

By using now the fact that

$$\int_{|z|<X_0, r<R} vKvd\nu \geq \alpha,$$

it follows then from (4.38) and (4.41) that

$$\int_{|z|<X_0, r<X_0} r^2 vd\nu + \sum_{i=1}^{i=n_*} \int_{|z|<X_0, X_{i-1}\leq r<X_i} vd\nu \geq \frac{\alpha}{m} X^{-\frac{k+1}{k}}.$$

This completes the proof. $\square$

4.5 Proofs of our main results

In this section, with all the estimates and properties of the operator K and the functional Φ_λ^n that have been derived in sections 4.3 and 4.4, we will be able to prove our main result. We begin with a Lemma giving a lower bound for the volume of the set

$$\{(r,z) \in \Pi | Kv(r,z) - \frac{\lambda}{2n} r^{2n} > 0\},$$

where $v \in L^1(\Pi, \nu) \cap L^p(\Pi, \nu)$ $(p > 1)$.

Lemma 4.15. *Let $\lambda > 0$, let $n \geq 1$, let $p \geq 1$ and let $v \in L^1(\Pi, \nu) \cap L^p(\Pi, \nu)$ be a non-negative function. Then for all $\lambda \in (0, \frac{nC(v)}{12\pi^2})$ we have*

$$\nu(\{(r,z) \in \Pi | Kv(r,z) - \frac{\lambda}{2n} r^{2n} > 0\}) \geq \frac{4\pi}{3}((\frac{nC(v)}{12\pi^2 \lambda})^{1/n} - 1),$$

where

$$C(v) = \int_{\Pi} \frac{r^2}{1 + |\rho|^3} v(r,z) d\nu \quad \textit{and} \quad \rho = (r,z).$$

Proof. By using Lemma 4.4, for all $(r,z) \in \Pi$ we have

$$Kv(r,z) \geq \frac{r^2}{24\pi^2(1 + |\rho|^3)} \int_{\Pi} \frac{r'^2}{1 + |\rho'|^3} v(r', z') d\nu'.$$

Then it follows that for all $n \geq 1$

$$Kv(r,z) \geq \frac{r^{2n} C(v)}{24\pi^2(1 + |\rho|^3)^n},$$

because $r^2 < 1 + |\rho|^3$. Therefore we have

$$\begin{aligned} \nu(\{(r,z) \in \Pi | Kv(r,z) - \frac{\lambda}{2n} r^{2n} > 0\}) &\geq \nu(\{|\rho|^3 < (\frac{nC(v)}{12\pi^2\lambda})^{1/n} - 1\}) \\ &= \frac{4\pi}{3}((\frac{nC(v)}{12\pi^2\lambda})^{1/n} - 1). \end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 4.1

Let $\lambda > 0$, let $n \geq 4$ be fixed, let $p > 5/2$, let q be the conjugate exponent of p, and let a be a positive number such that $\nu(\{(r,z) \in \Pi | v_0(r,z) > 0\}) = 2\pi^2 a^3$. Let $\mathcal{F}(v_0)$ be the set of all rearrangements of v_0 on Π. Let $\mathcal{R}(v_0)$ be the set defined as in Lemma 4.9, and let $\mathcal{R}^s(v_0)$ be the set of all Steiner-symmetric functions in $\mathcal{R}(v_0)$. Let X be a positive number chosen as in Lemma 4.9. By Lemma 4.10, a maximising sequence $\{v_j\}_{j=1}^{\infty}$ for Φ_λ^n relative to $\mathcal{F}(v_0)$ can be chosen in $\mathcal{R}(v_0)$. Hence if we assume that $v_j \in \mathcal{R}(v_0)$, then by Lemma 4.9, for each j, we can choose f_j having bounded support, for which we have $f_j \in \mathcal{R}(v_0)$, $\operatorname{supp} f_j \subset \mathbb{R} \times (0, X)$ and $\Phi_\lambda^n(f_j) \geq \Phi_\lambda^n(v_j)$.

Let us consider $\{\xi_j\}_{j=1}^{\infty}$ a sequence chosen for which $\xi_j \geq j$ and $\operatorname{supp} f_j \subset \Pi(\xi_j, X)$ for all $j \in \mathbb{N}$. Also define the sets

$$\mathcal{J}(\xi_j, X) = \{v \in \mathcal{R}(v_0) | \operatorname{supp} v \subset \Pi(\xi_j, X)\}.$$

Lemma 4.8 shows that the operator

$$K : L^p(\Pi(\xi_j, X), \nu) \to L^q(\Pi(\xi_j, X), \nu)$$

is symmetric, compact and strictly positive; hence the functional Φ_λ^n is weakly sequentially continuous and strictly convex on $L^p(\Pi(\xi_j, X), \nu)$. Therefore by using Theorem 1.1, the functional Φ_λ^n attains a maximum value relative to $\mathcal{J}(\xi_j, X)$. If v is a maximiser of Φ_λ^n relative $\mathcal{J}(\xi_j, X)$, then it follows from Riesz's rearrangement inequality (1.3) that

$$\Phi_\lambda^n(v^s) \geq \Phi_\lambda^n(v).$$

Thus, there exists a maximiser for the functional Φ_λ^n relative to $\mathcal{J}(\xi_j, X)$ and it may be chosen in the set

$$\mathcal{J}^s(\xi_j, X) = \{v \in \mathcal{R}^s(v_0) | \operatorname{supp} v \subset \Pi(\xi_j, X)\}.$$

For each $j \in \mathbb{N}$, let $\bar{v}_j$ be a maximiser for Φ_λ^n which belongs to the set $\mathcal{J}^s(j, X)$. From above we obtain

$$\Phi_\lambda^n(\bar{v}_j) \geq \Phi_\lambda^n(f_j),$$

hence $\{\bar{v}_j\}_{j=1}^{\infty}$ is a maximising sequence for the functional Φ_λ^n, and then by applying Theorem 1.1 again; there is an increasing function ϕ_j such that

$$\bar{v}_j = \phi_j \circ (K\bar{v}_j - \frac{\lambda}{2n} r^{2n}) \tag{4.42}$$

almost everywhere in $\Pi(\xi_j, X)$, for fixed λ and n.

We are now going to show that if $j \to \infty$, then the support of $\bar{v}_j$ is a subset of the set where $K\bar{v}_j - \frac{\lambda}{2n} r^{2n} > 0$; hence by using (4.42) we need just to show that the volume of the set where $K\bar{v}_j - \frac{\lambda}{2n} r^{2n} > 0$, is greater than $2\pi^2 a^3$. Indeed, by Lemma 4.11 we can choose some positive constant α as

$$\alpha = \Phi_\lambda^n(f) > 0$$

for some function $f \in \mathcal{R}(v_0)$. Hence there is a positive number $j_0 \in \mathbb{N}$, which may depend on λ such that

$$\Phi_\lambda^n(\bar{v}_j) \geq \alpha$$

for all $j \geq j_0$ and $\lambda > 0$. Now let $\xi > 0$ be positive. By Lemma 4.12, there exists a positive number β that satisfies

$$\int\limits_{S(\xi)} |\bar{v}_j(r,z)|^p d\nu > \beta^p,$$

where $S(\xi)$ is some square of side ξ. We can assume $S(\xi)$ is symmetric in the r-axis because $\bar{v}_j$ is symmetric. It follows therefore from Lemma 4.13 that there exists a positive number $\eta > 0$ such that

$$\int\limits_{S(\xi)} r^2 \bar{v}_j(r,z) d\nu \geq \eta \tag{4.43}$$

for all $j \geq j_0$. Setting

$$C(\xi) = \min_{S(\xi)}\left(\frac{1}{1+(r^2+z^2)^{3/2}}\right) \geq \frac{1}{1+(\xi^2/4+X^2)^{3/2}},$$

then from (4.43) and Lemma 4.4, we get

$$K\bar{v}_j(r,z) \geq \frac{C(\xi)\eta r^2}{24\pi^2(1+(r^2+z^2)^{3/2})}. \tag{4.44}$$

For all $(r,z) \in \Pi$, we can show that

$$1+(r^2+z^2)^{3/2} \leq 8(1+r^3)(1+|z|^3). \tag{4.45}$$

Combine (4.44) with (4.45) we have

$$K\bar{v}_j(r,z) \geq \frac{Nr^2}{(1+r^3)(1+|z|^3)},$$

where $N = \frac{C(\xi)\eta}{384\pi^2}$. Therefore it follows that

$$K\bar{v}_j(r,z) - \frac{\lambda}{2n} r^{2n} \geq \left(\frac{N}{(1+r^3)(1+|z|^3)} - \frac{\lambda}{2n} r^{2n-2}\right) r^2.$$

Now the region defined by the inequality

$$|z| < \left(\frac{2Nn}{\lambda(1+r^3)r^{2n-2}} - 1\right)^{1/3} \quad \text{and} \quad r < X$$

has infinite volume with respect to ν measure because $n \geq 4$. Hence we can choose $\varepsilon > 0$ independent of j such that

$$\nu(\{(r,z) \in \mathbb{R} \times (0,X) | |z|^3 < \frac{2Nn}{(1+r^3)(n\varepsilon + \lambda r^{2n-2})} - 1\}) \geq 2\pi^2 a^3. \tag{4.46}$$

We set

$$V(\xi_j) = \{(r,z) \in \Pi | \bar{v}_j(r,z) > 0\},$$

and

$$R(\xi_j, X) = \{(r,z) \in \Pi(\xi_j, X) | K\bar{v}_j(r,z) - \frac{\lambda}{2n} r^{2n} \geq \varepsilon\}.$$

Since $j \to \infty$, then from (4.46) we conclude that $\nu(R(\xi_j, X)) \geq 2\pi^2 a^3$. Now the fact that $\bar{v}_j$ is an increasing function of $K\bar{v}_j(r,z) - \frac{\lambda}{2n} r^{2n}$ and (4.42) implies that apart from a set of zero measure

$$V(\xi_j) \subset R(\xi_j, X).$$

Lemma 4.7 shows that all points $(r,z) \in R(\xi_j, X)$ satisfy

$$M' \|v_0\|_p (r^{1+\frac{1}{k}} + r^{1+\frac{1}{2k}-\frac{1}{q}}) \min\{1, |z|^{\frac{-1}{2k+p}}\} - \frac{\lambda}{2n} r^{2n} \geq \varepsilon,$$

where M' is a positive constant independent of j. Thus there exists a positive number Z independent of j such that $|z| < Z$ for all $(r,z) \in \operatorname{supp} \bar{v}_j$. We set $\bar{\xi} = \max\{\xi, Z\}$ and let j_1 be an integer such that $\xi_j > \bar{\xi}$ for all $j \geq j_1$; then the support of $\bar{v}_j$ is bounded by the rectangle $\Pi(\bar{\xi}, X)$, so $\bar{v}_j \in \mathcal{J}(\xi_{j^*}, X)$ for all $j \geq j^*$, where $j^* = \max\{j_0, j_1\}$. Therefore $\bar{v}_j$ maximises the functional Φ_λ^n relative to $\mathcal{J}(\xi_j, X)$ for all $j \geq j^*$. Thus $\bar{v}_j$ maximises the functional Φ_λ^n relative to $\mathcal{R}(v_0)$. Therefore $\bar{v}_j$ maximises Φ_λ^n relative to $\mathcal{F}(v_0)$. By writing $\bar{v}_j(r,z) = v(r,z)$, then from (4.42) we get

$$v = \phi_{j_*} \circ (Kv - \frac{\lambda}{2n} r^{2n}) \tag{4.47}$$

almost everywhere in $\Pi(\bar{\xi}, X)$. It remains only to extend (4.47) to $\Pi \backslash \Pi(\bar{\xi}, X)$. We can assume that $\phi_{j^*}(t) \geq 0$ for all $t \in \operatorname{dom} \phi_{j_*}$ and we consider the function ϕ defined by

$$\phi(t) = \begin{cases} \phi_{j^*}(t) & \text{if } t > \varepsilon, \\ 0 & \text{if } t \leq \varepsilon. \end{cases}$$

Since $Kv(r,z) - \frac{\lambda}{2n} r^{2n} \leq \varepsilon$ outside $\Pi(\bar{\xi}, X)$ and ϕ_{j^*} is an increasing function of $Kv(r,z) - \frac{\lambda}{2n} r^{2n}$ almost everywhere in $\Pi(\bar{\xi}, X)$, then ϕ is an increasing function on Π. Hence

$$v = \phi \circ (Kv - \frac{\lambda}{2n} r^{2n})$$

almost everywhere in Π. Therefore by setting $\Psi := Kv$ we have

$$\mathcal{L}\Psi = \phi \circ (\Psi - \frac{\lambda}{2n} r^{2n})$$

almost everywhere in Π for some increasing function ϕ, λ positive and $n \geq 4$. This completes the proof in the case when $n \geq 4$ and λ is positive. $\square$

4.6 Discussion about the Conjecture 4.2

Henceforth, in this section we assume that $2 \leq n < 4$, X is the positive number provided by Lemma 4.9 and $k > \frac{2(9-4\tau)}{18n-33-4(2n-3)\tau}$, where $\tau = (\frac{2}{3})^8$. In the above section we proved that the functional Φ_λ^n attains a maximum value relative to $\mathcal{J}(\xi_j, X)$ without any restriction on λ and n. Thus, if we consider $\{\bar{v}_j\}_{j=1}^{\infty}$ to be sequence of maximisers for the functional Φ_λ^n relative to $\mathcal{J}(\xi_j, X)$, then it only remains to show that when λ is small enough and ξ_j is large, the support of $\bar{v}_j$ is contained in the region where $K\bar{v}_j(r,z) - \frac{\lambda}{2n}r^{2n} > 0$. Thus we need to show that the volume of the set where $K\bar{v}_j(r,z) - \frac{\lambda}{2n}r^{2n} > 0$ has a volume greater than $2\pi^2 a^3$. To do that, let us consider λ_0 a positive number chosen so that

$$\alpha = \Phi_{\lambda_0}^n(v_1) > 0,$$

where v_1 is some rearrangement of v_0 having bounded support in $\mathbb{R} \times (0, X)$. Then there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ and $0 < \lambda < \lambda_0$ we have

$$\Phi_\lambda^n(\bar{v}_j) \geq \alpha.$$

It then follows that

$$\int\limits_{r<R} \bar{v}_j(r,z) K\bar{v}_j(r,z) d\nu \geq 2\alpha \tag{4.48}$$

for all $j \geq j_0$ and $0 < \lambda < \lambda_0$. Now for $\epsilon > 0$ independent of X and j, we can write

$$\int\limits_{r<X, |z|<\epsilon X} \bar{v}_j(r,z) K\bar{v}_j(r,z) d\nu \geq 2\alpha - \int\limits_{r<X, |z|\geq\epsilon X} \bar{v}_j(r,z) K\bar{v}_j(r,z) d\nu. \tag{4.49}$$

In Chapter 2, section 6, we gave an alternative proof of Theorem 2.1 when $n = 2$; this approach will be used as starting point to show that we have a chance to make Conjecture 4.2 into a Theorem. Then by following the same strategy that has been used in Chapter 2, section 6, we find the most important step to prove Conjecture 4.2, is to prove that the right hand in (4.49) has a positive lower bound independent of X. In other words, we need to prove that

$$\int\limits_{r<X, |z|\geq X} \bar{v}_j(r,z) K\bar{v}_j(r,z) d\nu \leq \beta, \tag{4.50}$$

where β is a positive number that may depend on X, although if it does, then it must be bounded. Otherwise, we find that the left hand in (4.50) tends to ∞ when $X \to \infty$, and that means that the support of $\bar{v}_j$ becomes very thin in the r-direction and very long in z-direction, but this is an unlikely shape for the maximiser to take. Now Lemma 4.7 does not provide enough justification for (4.50), because when we use it, we find the number β depends on X and tends to ∞. Therefore, we conclude that, to prove the Conjecture 4.2, it would be better to prove (4.50) by different means. To make a justification for (4.50), it would be sufficient to obtain an estimate such as

$$Kv(r,z) \leq C \min\{1, |z|^{-N}\} f(r), \tag{4.51}$$

where $N > 0$, C is a positive constant that may depend on $\|v\|_p$ and f is a positive function such that $X^{-N} f(X)$ is bounded independent of X. Henceforth, we assume that $0 < \lambda < \lambda_0$, $j \geq j_0$ and the estimate (without proof) in (4.51) holds. Since ϵ is arbitrary, then we can choose $\epsilon > 0$ independent of λ and j so that (4.50) holds in the form

$$\int\limits_{r<X,|z|\geq \epsilon X} \bar{v}_j(r,z) K\bar{v}_j(r,z) d\nu \leq \alpha.$$

Hence by using (4.49) we get

$$\int\limits_{r<X,|z|<\epsilon X} \bar{v}_j(r,z) K\bar{v}_j(r,z) d\nu \geq \alpha. \tag{4.52}$$

From the proof of Lemma 4.9 we deduce that

$$X = (\frac{2nM\|v_0\|_p}{\lambda})^{\frac{k}{(2n-1)k-1}},$$

where M is a positive constant independent of λ and j. Now for all $i \in \{0, 9\}$, we define the sequence X_i as follows

$$X_i = \begin{cases} (\frac{2nM\|v_0\|_p}{\lambda})^{t_i} & \text{if } i \leq 8 \\ X & \text{if } i = 9, \end{cases}$$

where

$$t_i = t_0(3 - 2(\frac{2}{3})^i) \quad \text{and} \quad t_0 = \frac{3k}{((2n-1)k-1)(9-4\tau)}.$$

Hence by comparing X with X_0, we find that

$$X_0 = X^{\frac{3}{9-4\tau}}. \tag{4.53}$$

Now let λ_1 be positive number, which may depend on ϵ, chosen so that $\epsilon X > X_0$ when $0 < \lambda < \lambda_1$. Since $\bar{v}_j$ is Steiner-symmetric, then it follows that

$$\int_{r<X,|z|<\epsilon X} \bar{v}_j(r,z) K\bar{v}_j(r,z) d\nu \leq \frac{\epsilon X}{X_0} \int_{r<X,|z|<X_0} \bar{v}_j(r,z) K\bar{v}_j(r,z) d\nu.$$

By applying (4.52) and (4.53) we have

$$\int_{r<X,|z|<X_0} \bar{v}_j(r,z) K\bar{v}_j(r,z) d\nu \quad \geq \quad \frac{X_0}{\epsilon X} \int_{r<X,|z|<\epsilon X} \bar{v}_j(r,z) K\bar{v}_j(r,z) d\nu$$

$$
\begin{aligned}
&= \frac{X^{\frac{4\tau-6}{9-4\tau}}}{\epsilon} \int_{r<X,|z|<\epsilon X} \bar{v}_j(r,z) K \bar{v}_j(r,z) d\nu \\
&\geq \frac{\alpha}{\epsilon} X^{\frac{4\tau-6}{9-4\tau}}. \qquad (4.54)
\end{aligned}
$$

Therefore from Lemma 4.14 and (4.54) we have

$$
\int_{|z|<X, r<X_0} r^2 \bar{v}_j(r,z) d\nu + \sum_{i=1}^{i=9} \int_{|z|<X_0, X_{i-1}\leq r<R_i} \bar{v}_j(r,z) d\nu \geq \frac{\alpha}{\epsilon m} X^{\frac{(4\tau-6)k-k-1}{(9-4\tau)k}}, \qquad (4.55)
$$

where m is a positive constant independent of λ and j. Hence by using the form of X we get

$$
\int_{|z|<X_0, r<X_0} r^2 \bar{v}_j(r,z) d\nu + \sum_{i=1}^{i=9} \int_{|z|<X_0, X_{i-1}\leq r<X_i} \bar{v}_j(r,z) d\nu \geq C \lambda^{\frac{(15-8\tau)k+(9-4\tau)}{((2n-1)k-1))(9-4\tau)}}, \qquad (4.56)
$$

where C is a positive constant depending on α, ϵ and k and m. We set now

$$
C(\bar{v}_j) = \int_{r<X} \frac{r^2}{1+|\rho|^3} \bar{v}_j(r,z) d\nu,
$$

where $|\rho| = (r^2+z^2)^{1/2}$. Then we have

$$
C(\bar{v}_j) \geq \left(\int_{|z|<X_0, r<X_0} + \sum_{i=1}^{i=9} \int_{|z|<X_0, X_{i-1}\leq r<X_i} \right) \frac{r^2}{1+|\rho|^3} \bar{v}_j(r,z) d\nu. \qquad (4.57)
$$

If $|z| < X_0$ and $r < X_0$, then

$$
1+|\rho|^3 \leq 2 (\frac{2nM\|v_0\|_p}{\lambda})^{3t_0}.
$$

Thus we find that

$$
\int_{|z|<X_0, r<X_0} \frac{r^2}{1+|\rho|^3} \bar{v}_j(r,z) d\nu \geq \frac{1}{2} (\frac{\lambda}{2nM\|v_0\|_p})^{3t_0} C_1(\bar{v}_j), \qquad (4.58)
$$

where

$$
C_1(\bar{v}_j) = \int_{|z|<X_0, r<X_0} r^2 \bar{v}_j(r,z) d\nu.
$$

Also, if $|z| < X_0$ and $r < X_i$, then

$$
1+|\rho|^3 \leq 2 (\frac{2nM\|v_0\|_p}{\lambda})^{3t_i}
$$

provided that $0 < \lambda < \min\{\lambda_0, \lambda_1\}$. Hence it follows that

$$\int\limits_{|z|<X_0, X_{i-1}\leq r<X_i} \frac{r^2}{1+|\rho|^3}\bar{v}_j(r,z)d\nu \geq \frac{1}{2}(\frac{\lambda}{2nM\|v_0\|_p})^{3t_i-2t_{i-1}}C_2(\bar{v}_j), \tag{4.59}$$

where

$$C_2(\bar{v}_j) = \int\limits_{|z|<X_0, X_{i-1}\leq r<X_i} \bar{v}_j(r,z)d\nu.$$

Now for all $i \in \{0, 9\}$, t_i satisfies

$$3t_0 = 3t_1 - 2t_0 = \cdots = 3t_i - 2t_{i-1} = \cdots = \frac{3k}{(2n-1)k-1} - 2t_8.$$

Hence from (4.59) we get

$$\int\limits_{|z|<X_0, X_{i-1}\leq r<X_i} \frac{r^2}{1+|\rho|^3}\bar{v}_j(r,z)d\nu \geq \frac{1}{2}(\frac{\lambda}{2nM\|v_0\|_p})^{3t_0}C_2(\bar{v}_j). \tag{4.60}$$

It follows from (4.56), (4.57), (4.58) and (4.60)

$$C(\bar{v}_j) \geq A\lambda^{\frac{(24-8\tau)k+(9-4\tau)}{((2n-1)k-1)(9-4\tau)}}, \tag{4.61}$$

where $A = \frac{C}{2(2nM\|v_0\|_p)^{3t_0}}$, and then from (4.61) we deduce that

$$\frac{C(\bar{v}_j)}{\lambda} \geq A(\frac{1}{\lambda})^{\frac{((18n-33)-4(2n-3)\tau)k-2(9-4\tau)}{((2n-1)k-1)(9-4\tau)}}. \tag{4.62}$$

Therefore by applying Lemma 4.15 and (4.62) we have

$$\nu(\{(r,z) \in \Pi(\xi_j, X) | K\bar{v}_j(r,z) - \frac{\lambda}{2n}r^{2n} > 0\}) \geq F(\lambda),$$

where

$$F(\lambda) = \frac{4\pi}{3}((\frac{nA}{12\pi^2})^{1/n}(\frac{1}{\lambda})^{\frac{((18n-33)-4(2n-3)\tau)k-2(9-4\tau)}{n((2n-1)k-1))(9-4\tau)}} - 1)$$

provided $j \geq j_0$ and $0 < \lambda < \min\{\lambda_0, \lambda_1\}$. Since A is independent of λ and j, $2 \leq n < 4$ and $k > \frac{2(9-4\tau)}{(18n-33)-4(2n-3)\tau}$, then it follows that $F(\lambda) \to \infty$ as $\lambda \to 0$. Thus we can choose $\varepsilon > 0$ independent of j and λ_2 independent of j and λ such that if $0 < \lambda < \Lambda = \min\{\lambda_0, \lambda_1, \lambda_2\}$ and $\xi_j \to \infty$ as $j \to \infty$, then we have

$$\nu(\{\frac{A\lambda^{\frac{(24-8\tau)k+(9-4\tau)}{((2n-1)k-1)(9-4\tau)}}r^2}{24\pi^2(1+|\rho|^3)} - \frac{\lambda}{4}r^4 \geq \varepsilon\}) \geq 2\pi^2a^3.$$

It follows then that the set

$$R(\xi_j, X) = \{(r,z) \in \Pi(\xi_j, X) | K\bar{v}_j(r,z) - \frac{\lambda}{2n} r^{2n} \geq \varepsilon\}$$

has volume at least $2\pi^2 a^3$ provided $\lambda < \Lambda = \{\lambda_0, \lambda_1, \lambda_2\}$ and $\xi_j \to \infty$ as $j \to \infty$. Therefore as in the case when $n \geq 4$ we have

$$V(\xi_j) \subset R(\xi_j, X).$$

Now Lemma 4.7 shows that all points $(r,z) \in V(\xi_j)$ satisfy

$$M' \|v_0\|_p (r^{1+\frac{1}{k}} + r^{1+\frac{1}{2k}-\frac{1}{q}}) \min\{1, |z|^{\frac{-1}{2k+p}}\} - \frac{\lambda}{2n} r^{2n} \geq \varepsilon,$$

where M' is a positive constant independent of j. Therefore there exists a positive constant Z independent of j such that $|z| < Z$. Now for $0 < \lambda < \Lambda$ we set $r(\lambda) = \max\{X, Z\}$ and let $j_* \in \mathbb{N}$ be such that $\xi_j \geq r(\lambda)$ for all $j \geq j_*$. Then by following the same procedure that was used in the case when $n \geq 4$ we find that $\bar{v}_{j_*}$ maximises the functional Φ_λ^n relative to $\mathcal{R}(v_0)$ and all positive small λ, and therefore $\bar{v}_j$ maximises the functional Φ_λ^n relative to $\mathcal{F}(v_0)$. Hence by writing $\bar{v}_{j_*} = v$ and $\Psi := Kv$ we have

$$\mathcal{L}\Psi = \phi \circ (\Psi - \frac{\lambda}{2n} r^{2n})$$

almost everywhere in Π for some increasing function, in the case when $\lambda > 0$ is small and $2 \leq n < 4$.

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